# QUANTUM AFFINE KNIZHNIK-ZAMOLODCHIKOV EQUATIONS AND QUANTUM SPHERICAL FUNCTIONS, I

JASPER V. STOKMAN

Abstract. Cherednik's quantum affine Knizhnik-Zamolodchikov equations associated to an affine Hecke algebra module $M$ form a holonomic system of $q$-difference equations acting on $M$-valued functions on a complex torus $T$. In this paper the quantum affine Knizhnik-Zamolodchikov equations are related to the Cherednik-Macdonald theory when $M$ is induced from a character of a standard parabolic subalgebra of the affine Hecke algebra. We set up correspondences between solutions of the quantum affine KZ equations and, on the one hand, solutions to the spectral problem of the Cherednik-Dunkl $q$-difference reflection operators (generalizing work of Kasatani and Takeyama) and, on the other hand, solutions to the spectral problem of the Cherednik-Macdonald $q$-difference operators (generalizing work of Cherednik). The correspondences are applicable to all relevant spaces of functions on $T$ and for all parameter values, including the cases that $q$ and/or the Hecke algebra parameters are roots of unity.

Contents



## 1. Introduction

This is the first part of a sequel of papers reporting on the analysis of Cherednik's [2, 4] quantum affine Knizhnik-Zamolodchikov (KZ) equations and their applications to quantum harmonic analysis and integrable systems.

Cherednik's [2, 4] quantum affine KZ equations associated to an affine Hecke algebra module $M$ form a holonomic system of first order $q$-difference equations (an integrable $q$-connection) acting on $M$-valued functions on a complex torus $T$; typically meromorphic or Laurent polynomial solutions are considered. For $0 < |q| < 1$ and for $M$ a principal series module (i.e. obtained from inducing a character of the minimal standard parabolic subalgebra of the affine Hecke algebra), Cherednik [4]

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studied a correspondence between meromorphic solutions of the associated quantum affine KZ equations on the one hand, and common meromorphic eigenfunctions of the Cherednik-Macdonald $q$-difference operators on the other hand. One of the objectives of the present paper is to determine explicit conditions on the induction data of the principal series module to ensure the bijectivity of this correspondence.

In addition we generalize and refine Cherednik's correspondence, allowing modules $M$ that are induced from a character of a standard parabolic subalgebra of the affine Hecke algebra, allowing arbitrary classes of functions on $T$, and allowing all values of the quantum parameter $q$ and of the Hecke algebra parameters $k$ (including roots of unity). The main properties of this generalized and refined correspondence are stated in Theorem 5.16.

The classical analogue of Cherednik's correspondence is due to Matsuo [25] and Cherednik [3]. It was pursued further by Opdam [28] and by Cherednik and Ma [9]. A substantial part of the present work is inspired by Opdam's [28, §3] approach to the classical correspondence, in which Cherednik's trigonometric analogues of Dunkl's differential-reflection operators naturally come into play. In the present quantum setup, Opdam's approach suggests a natural intermediate stage of the correspondence in which solutions of quantum affine KZ equations are related to common eigenfunctions of the Cherednik-Dunkl commuting $q$-difference reflection operators. Such a correspondence has indeed recently been established for root system of type A by Kasatani and Takeyama [17]. We extend these results to arbitrary root systems in Theorem 4.9.

For $q = 1$, for root system of type A, and for $M$ induced from a character a maximal standard parabolic subalgebra, the $q$-connection matrices of the quantum KZ equations are interpolants of the transfer matrix of an inhomogeneous spin chain. It leads to the possibility to construct particular eigenstates for XXZ spin chains from solutions of quantum affine KZ equations. In the case that the Hecke algebra parameter is a third root of unity this approach is explored extensively in the context of the Razumov-Stroganov conjectures (see, e.g., [31, 30, 12, 17, 16]). The correspondences investigated in the present paper, in case that the underlying root system is of classical type, are expected to be useful in the analysis of recent generalizations [13, 14] of the Razumov-Stroganov conjectures.

Part II of the present paper will be devoted to the interplay between quantum affine KZ equations and quantum harmonic analysis. The crucial starting point will be the fact that for $0 < |q| < 1$, asymptotically free solutions of the quantum KZ equations can be constructed which map, under the correspondence, to $q$-analogues of the Harish-Chandra series (see [27, 26]). The results in the present paper then lead to explicit conditions on the spectral parameters to ensure that the $q$-analogues of the Harish-Chandra series become a basis of the meromorphic solution space of the spectral problem of the Cherednik-Macdonald $q$-difference operators. Combined with the recent results of Cherednik [8] on $q$-analogues of the Harish-Chandra $c$-function, this will accumulate in the derivation of an explicit expansion of Cherednik's [5] quantum spherical function in terms of the $q$-analogues of the Harish-Chandra series (generalizing Harish-Chandra's $c$-function expansion of the spherical function).

**Conventions**

Lots of results come in two forms: a "+"-version, related to symmetric theory, and a "−"-version, related to antisymmetric theory. We formulate both versions at the

same time, labeling the objects by $\pm$. An equality like $\pm a = \mp b$ should thus be read as $a = -b$ and $-a = b$; it will always be clear from context which of the two equalities should be seen as the $+$-version and which as the $-$-version.

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## 2. The classes of affine Hecke algebra modules

In this section we recall the definition of the affine Hecke algebra. In addition, we discuss the affine Hecke algebra modules obtained by inducing a suitable character of a standard parabolic subalgebra.

2.1. **Weyl groups and parabolic subgroups.** Let $R_0 \subset V_0$ be a finite, crystallographic, reduced, irreducible root system in an Euclidean space $\big(V_0, \langle\cdot,\cdot\rangle\big)$ of dimension $n$. We normalize the roots in such a way that long roots have squared length 2. The Weyl group of $R_0$ is denoted by $W_0$. We fix a basis $\Delta_0 = \{\alpha_1, \ldots, \alpha_n\}$ of $R_0$ once and for all. Let $R_0^{\pm}$, $\varphi$, $\{s_i\}_{i=1}^n$, $w_0$ be the associated positive/negative roots, longest root, simple reflections and longest Weyl group element, respectively. The length of a Weyl group element $w \in W_0$ is $l(w) := \#(R_0^+ \cap w^{-1}R_0^-)$. The associated Bruhat ordering on $W_0$ is denoted by $\leq$.

Unless specified explicitly otherwise, $I$ will always stand for a fixed subset of $\{1, \ldots, n\}$. We write $W_{0,I}$ for the subgroup of $W_0$ generated by the simple reflections $s_i$ ($i \in I$). Then $R_0^I = R_0 \cap \mathrm{span}_{\mathbb{Z}}\{\alpha_i\}_{i\in I}$ is a root system in $V_{0,I} := \bigoplus_{i\in I} \mathbb{R}\alpha_i$ with Weyl group isomorphic to $W_{0,I}$. Furthermore, $\{\alpha_i\}_{i\in I}$ is a basis of $R_0^I$. We write $R_0^{I,\pm}$ for the associated set of positive and negative roots, respectively. The length function on $W_{0,I}$ coincides with the restriction of the length function $l$ to $W_{0,I}$.

Set
$$
\begin{aligned}
W_0^I :=&\{w \in W_0 \mid l(ws_i) > l(w) \quad \forall\, i \in I\}\\
=&\{w \in W_0 \mid w(R_0^{I,+}) \subseteq R_0^+\}.
\end{aligned}
$$
It is a complete set of representatives of the coset space $W_0/W_{0,I}$. Furthermore,
$$
l(uv) = l(u) + l(v), \qquad \forall\, u \in W_0^I,\, \forall\, v \in W_{0,I}.
$$
The elements of $W_0^I$ are called the minimal coset representatives of $W_0/W_{0,I}$. When decomposing an element $w \in W_0$ as a product of a minimal coset representative and an element of $W_{0,I}$, we use the notation
$$
w = \overline{w}\underline{w} \qquad (\overline{w} \in W_0^I,\, \underline{w} \in W_{0,I})
$$
(although $\overline{w}$ and $\underline{w}$ depends on the choice of $I$, we suppress this from the notations). We will frequently make use of the following elementary lemma (see [10, Lemma 2.1]).

**Lemma 2.1.** *Fix $1 \leq i \leq n$. Define*
$$
\begin{aligned}
A_i &= \{w \in W_0^I \mid l(s_iw) = l(w) - 1\},\\
B_i &= \{w \in W_0^I \mid l(s_iw) = l(w) + 1 \ \&\ s_iw \in W_0^I\},\\
C_i &= \{w \in W_0^I \mid l(s_iw) = l(w) + 1 \ \&\ s_iw \notin W_0^I\}.
\end{aligned}
$$

*Then*

**(i)** $W_0^I = A_i \cup B_i \cup C_i$ *(disjoint union).*
**(ii)** *The map $w \mapsto s_i w$ defines an involution of $A_i \cup B_i$. It maps $A_i$ onto $B_i$.*
**(ii)** *For $w \in C_i$ there exists a unique $i_w \in I$ such that $s_i w = w s_{i_w}$. In particular, $\overline{s_i w} = w$.*

We record here a technical lemma, which we will be needing at a later stage.

**Lemma 2.2.** *Suppose $w \in W_0^I$ and $w \notin B_i$ for all $1 \leq i \leq n$. Then $w = \overline{w_0}$.*

*Proof.* Suppose to the contrary that $w \neq \overline{w_0}$. Then $w\underline{w_0} \neq w_0$, hence there exists an $i \in \{1, \ldots, n\}$ such that

$$l(s_i w \underline{w_0}) = l(w\underline{w_0}) + 1 = l(w) + l(\underline{w_0}) + 1. \tag{2.1}$$

If $s_i w \in W_0^I$ then it follows from (2.1) that $l(s_i w) = l(w) + 1$, contradicting the fact that $w \notin B_i$. If $s_i w \notin W_0^I$ then $w \in C_i$, hence $s_i w = w s_{i_w}$ with $i_w \in I$. Consequently

$$l(s_i w \underline{w_0}) = l(w s_{i_w} \underline{w_0}) = l(w) + l(s_{i_w} \underline{w_0}) = l(w) + l(\underline{w_0}) - 1,$$

which contradicts (2.1). □

2.2. **Affine root systems and affine Weyl groups.** We recall in this subsection some facts on twisted affine root systems. We stick as much as possible to the treatment in [23, Chpt. 1& 2].

Identify $V := \mathbb{R}c \oplus V_0$ with the real vector space of affine linear, real valued functionals on $V_0$ by interpreting $rc + v$ as the functional $v' \mapsto r + \langle v, v' \rangle$. Let $D : V \to V_0$ be the projection onto $V_0$ along the direct sum decomposition $V = \mathbb{R}c \oplus V_0$. We extend the scalar product $\langle \cdot, \cdot \rangle$ to a symmetric bilinear form on $V$ by requiring $D$ to be form preserving. The correponding semi-norm on $V$ is denoted by $\| \cdot \|$. We write $a^\vee = 2a/\|a\|^2$ for a vector $a \in V$ satisfying $\|a\| \neq 0$.

The twisted affine root system associated to $R_0$ is the set

$$R := \{a^\vee \mid a \in \mathbb{Z}c + R_0\} \subset V.$$

Let $W^a \subset \mathrm{O}(V)$ (with $\mathrm{O}(V)$ the group of invertible linear endomorphisms of $V$ preserving the symmetric bilinear form $\langle \cdot, \cdot \rangle$) be the subgroup generated by the involutions

$$s_a : v \mapsto v - \langle a, v \rangle a^\vee, \qquad v \in V$$

for all $a \in R$. It is called the affine Weyl group of $R$.

The affine Weyl group $W^a$ admits two important alternative descriptions, namely as the semidirect product group $W_0 \ltimes Q^\vee$ associated to the ($W_0$-invariant) coroot lattice $Q^\vee = \mathrm{span}_{\mathbb{Z}}\{\alpha^\vee \mid \alpha \in R_0\}$ of $V_0$, and as a Coxeter group. Concretely, denote by $\tau : V_0 \to \mathrm{O}(V)$ the semi-norm preserving action of $V_0$ on $V$ given by

$$\tau(v)(rc + v') = (r - \langle v, v' \rangle)c + v'$$

for $v, v' \in V_0$ and $r \in \mathbb{R}$. Then $w\tau(v)w^{-1} = \tau(wv)$ for $w \in W_0$ and $v \in V_0$. Furthermore, $s_a = s_\alpha \tau(r\alpha^\vee)$ for $a = (rc+\alpha)^\vee$ ($r \in \mathbb{Z}$, $\alpha \in R_0$) and $W_0 \ltimes Q^\vee \simeq W^a$ by $(w, \lambda) \mapsto w\tau(\lambda)$. It follows from this description of $W^a$ that $R$ is $W^a$-invariant.

In fact, $R$ is invariant under the action of the extended affine Weyl group

$$W := \{w\tau(\lambda)\}_{w \in W_0, \lambda \in P^\vee} \simeq W_0 \ltimes P^\vee,$$

where $P^\vee := \{\lambda \in V_0 \mid \langle \lambda, \alpha\rangle \in \mathbb{Z}\ \forall\, \alpha \in Q\}$ is the coweight lattice of $R_0$ in $V_0$. We write $\{\varpi_i^\vee\}_{i=1}^n \subset P^\vee$ for the fundamental coweights with respect to the ordered basis $\Delta_0$ of $R_0$ (they are characterized by $\langle \varpi_i^\vee, \alpha_j\rangle = \delta_{i,j}$ for all $1 \leq j \leq n$).

Observe that the affine Weyl group $W^a$ is a normal subgroup of $W$ with finite abelian quotient group $W/W^a \simeq P^\vee/Q^\vee$.

We recall now the presentation of $W^a$ as a Coxeter group. An ordered basis $\Delta$ of the twisted affine root system $R^\vee$ is given by

$$\{a_0, a_1, \ldots, a_n\} := \{c - \varphi^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee\} \tag{2.2}$$

(recall that, by our convention that long roots in $R_0$ have squared length 2, we have $\varphi^\vee = \varphi$). Let $R^\pm$ be the associated sets of positive/negative roots. The corresponding simple reflections are denoted by $S := \{s_j := s_{a_j}\}_{j=0}^n$ (since for $1 \leq i \leq n$, $s_i|_{V_0} \in W_0$ is the simple reflection associated to the basis element $\alpha_i \in R_0$ as defined in the previous subsection, there is no conflict in notation). Note that $s_0 = s_\varphi \tau(-\varphi^\vee)$. Then $(W^a, S)$ is a Coxeter group with associated set of simple reflections $S$. The defining relations of $W^a$ in terms of $S$ are

$$\begin{aligned} s_i s_j s_i \cdots &= s_j s_i s_j \cdots \qquad (m_{ij} \text{ terms on both sides}),\\ s_i^2 &= 1 \end{aligned} \tag{2.3}$$

for $0 \leq i, j \leq n$ with, for the first identity, the additional requirements that $i \neq j$ and that $s_i s_j \in W^a$ has finite order $m_{ij}$.

The length $l(w)$ of $w \in W$ with respect to the choice $\Delta$ of positive roots of $R$ is defined by

$$l(w) := \#\big(R^+ \cap w^{-1} R^-\big), \qquad w \in W. \tag{2.4}$$

Its restriction to $W_0$ coincides with the length function of $W_0$ as considered in the previous subsection. The subset $\Omega := \{w \in W \mid l(w) = 0\}$ is a subgroup of $W$, isomorphic to $P^\vee/Q^\vee$, and $W \simeq \Omega \ltimes W^a$. In fact, $\Omega$ permutes the simple roots of $R$. Hence an element $\omega \in \Omega$ gives rise to a permutation of the index set $\{0, \ldots, n\}$ of the simple reflections of $R$, which we again denote by $\omega$. Consequently $\omega(a_i) = a_{\omega(i)}$ and $\omega s_i \omega^{-1} = s_{\omega(i)}$ for $0 \leq i \leq n$ and $\omega \in \Omega$.

2.3. **The affine Hecke algebra.** The results on the affine Hecke algebra which we recall in this section to fix notations, are well known. We match the notations to [23, Chpt. 4] as much as possible.

A multiplicity function on $R$ is a map $k : R \to \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, denoted by $a \mapsto k_a$ ($a \in R$), which satisfies $k_{w(a)} = k_a$ for all $w \in W$ and $a \in R$. From now on, $k$ will always stand for a multiplicity function on $R$. We write $k_j := k_{a_j}$ for $0 \leq j \leq n$. We emphasize that we are *not* assuming $k_j$ to be generic, in particular, it may be a root of unity.

Note that $k_a = k_{Da}$, hence it is uniquely determined by its restriction $k|_{R_0^\vee}$ to a multiplicity function of the underlying finite root system $R_0^\vee$. In particular, the value $k_a$ only depends on the seminorm $\|a\|$ of $a \in R$, hence $k$ takes on at most two different values.

The affine Hecke algebra $H^a(k)$ is the unital, complex associative algebra generated by $T_i$ ($0 \leq i \leq n$) with defining relations

$$\begin{aligned} T_i T_j T_i \cdots &= T_j T_i T_j \cdots \qquad (m_{ij} \text{ terms on both sides}),\\ (T_i - k_i)(T_i + k_i^{-1}) &= 0 \end{aligned} \tag{2.5}$$

for $0 \leq i, j \leq n$ with, for the first identity, the additional requirements that $i \neq j$ and that $s_i s_j \in W^a$ has finite order $m_{ij}$.

Let $w \in W^a$ and fix a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_{l(w)}}$ $(0 \leq i_j \leq n)$. Then $T_w := T_{i_1} T_{i_2} \cdots T_{i_{l(w)}} \in H(k)$ is well defined (it is by definition the unit of $H(k)$ if $w = e$ is the identity element of $W$). The $T_w$ $(w \in W^a)$ form a $\mathbb{C}$-linear basis of $H(k)$. The subalgebra $H_0(k)$ of $H^a(k)$ generated by $T_i$ $(1 \leq i \leq n)$ is called the finite Hecke algebra. The $T_w$ $(w \in W_0)$ form a $\mathbb{C}$-linear basis of $H_0(k)$.

The finite abelian subgroup $\Omega$ of $W$ acts by algebra automorphisms on $H^a(k)$ by $\omega(T_j) = T_{\omega(j)}$ for all $0 \leq j \leq n$ $(\omega \in \Omega)$. The *extended* affine Hecke algebra is the corresponding smashed product algebra $H(k) := H^a(k) \# \Omega$. Recall that, as a complex vector space, $H(k) \simeq H^a(k) \otimes_{\mathbb{C}} \mathbb{C}[\Omega]$ with $\mathbb{C}[\Omega]$ the complex group algebra of $\Omega$. The algebra structure of $H(k)$ is then characterized by $(h \otimes \omega)(h' \otimes \omega') = h\omega(h') \otimes \omega\omega'$ for $h, h' \in H^a(k)$ and $\omega, \omega' \in \Omega$.

A reduced expression of an extended affine Weyl group element $w \in W$ is an expression of $w$ of the form $w = s_{i_1} s_{i_2} \cdots s_{i_{l(w)}} \omega$ for some $0 \leq i_j \leq n$ and for some $\omega \in \Omega$. Then $T_w := T_{i_1} T_{i_2} \cdots T_{i_{l(w)}} \otimes \omega \in H(k)$ is well defined, it reduces to the previous definition of $T_w$ in case $w \in W^a$, and $\{T_w\}_{w \in W}$ is a $\mathbb{C}$-linear basis of $H(k)$.

Consider the complex torus

$$T := \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{C}^\times) \tag{2.6}$$

of group homomorphisms $P^\vee \to \mathbb{C}^\times$. We write $t^\lambda \in \mathbb{C}^\times$ $(t \in T,\ \lambda \in P^\vee)$ for the evaluation of $t$ at $\lambda$. For $z \in \mathbb{C}^\times$ and $\alpha \in Q$ with $Q \subset V_0$ the root lattice of $R_0$, we write $z^\alpha \in T$ for the group homomorphism $P^\vee \ni \lambda \mapsto z^{\langle \lambda, \alpha \rangle}$.

The Weyl group $W_0$ acts on $P^\vee$, By transposition, it also acts on $T$. Note that for $\alpha \in R_0$ and $t \in T$,

$$s_\alpha t = z^\alpha t \quad \text{with} \quad z = t^{-\alpha^\vee} \in \mathbb{C}^\times. \tag{2.7}$$

Let $\mathbb{C}[T]$ be the algebra of regular functions on $T$. We write $e^\lambda$ $(\lambda \in P^\vee)$ for the canonical $\mathbb{C}$-basis of $\mathbb{C}[T]$, where $e^\lambda$ stands for the regular function $T \ni t \mapsto t^\lambda$. Note that $\mathbb{C}[T]$ is isomorphic to the group algebra $\mathbb{C}[P^\vee]$ of the coweight lattice $P^\vee$.

The action of $W_0$ on $\mathbb{C}[T]$, contragredient to the $W_0$-action on $T$, satisfies $w(e^\lambda) := e^{w\lambda}$ for $w \in W_0$ and $\lambda \in P^\vee$. It is an action by algebra automorphisms on $\mathbb{C}[T]$, hence it extends uniquely to an action by field automorphisms on the quotient field $\mathbb{C}(T)$ of $\mathbb{C}[T]$. In addition, by transposition of the $W_0$-action on $T$, the finite Weyl group $W_0$ acts on the algebra $\mathcal{O}(T)$ of analytic functions on $T$ (respectively the field $\mathcal{M}(T)$ of meromorphic functions on $T$) by algebra (respectively field) automorphisms. It results in the following inclusion of $W_0$-module algebras,

$$\mathbb{C}[T] \subset \mathbb{C}(T) \subset \mathcal{O}(T) \subset \mathcal{M}(T).$$

For all $\alpha^\vee \in R_0^\vee$ define $c^k_{\alpha^\vee} \in \mathbb{C}(T)$ by

$$c^k_{\alpha^\vee}(t) := \frac{k_{\alpha^\vee}^{-1} - k_{\alpha^\vee} t^{\alpha^\vee}}{1 - t^{\alpha^\vee}}. \tag{2.8}$$

We furthermore write $c_i^k = c^k_{\alpha_i^\vee}$ for $1 \leq i \leq n$.

Let $P_+^\vee = \{\lambda \in P^\vee \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}\}$ be the cone of dominant coweights and set

$$Y^\lambda := T_{\tau(\lambda)} \in H(k).$$

Then $P_+^\vee \ni \lambda \mapsto Y^\lambda \in H(k)^\times$ is a morphism of semigroups. It has a unique extension to a group homomorphism $P^\vee \to H(k)^\times$, also denoted by $\lambda \mapsto Y^\lambda$ ($\lambda \in P^\vee$). Denote by $\mathcal{A}_Y^k$ the commutative subalgebra of $H(k)$ generated by the $Y^\lambda$ ($\lambda \in P^\vee$). The following theorem is due to Bernstein and Zelevinsky (for a proof, see [20] or [23]).

**Theorem 2.3.** **(i)** *The surjective algebra map $\mathbb{C}[T] \to \mathcal{A}_Y^k$ mapping $e^\lambda$ to $Y^\lambda$ for $\lambda \in P^\vee$, is an isomorphism. We write $f(Y)$ for the element in $\mathcal{A}_Y^k$ corresponding to $f \in \mathbb{C}[T]$ under this isomorphism.*
**(ii)** *For all $f \in \mathbb{C}[T]$ and $1 \leq i \leq n$,*

$$f(Y)T_i = T_i(s_i f)(Y) + (c_i^k(Y^{-1}) - k_i)((s_i f)(Y) - f(Y)), \tag{2.9}$$

*where $(c_i^k(Y^{-1}) - k_i)((s_i f)(Y) - f(Y))$ is the element of $\mathcal{A}_Y^k$ corresponding, under the isomorphism of* **(i)**, *to the regular function $(c_i^k(t^{-1}) - k_i)((s_i f)(t) - f(t))$ in $t \in T$.*
**(iii)** *The multiplication map defines an isomorphism $\mathcal{A}_Y^k \otimes_{\mathbb{C}} H_0(k) \xrightarrow{\sim} H(k)$ of vector spaces.*
**(iv)** *The cross relations* (2.9) *characterize the algebraic structure of $H(k)$ in terms of the algebras $\mathcal{A}_Y^k$ and $H_0(k)$.*

Recall that $I$ is a fixed subset of $\{1, \ldots, n\}$. Write $H_I(k)$ for the unital subalgebra of $H(k)$ generated by $\mathcal{A}_Y^k$ and the $T_i$ ($i \in I$). The subalgebra $H_{0,I}(k)$ generated by the $T_i$ ($i \in I$) has as complex linear basis $\{T_w\}_{w \in W_{0,I}}$. Then $H_I(k) \simeq \mathbb{C}_Y[T] \otimes_{\mathbb{C}} H_{0,I}(k)$ as vector spaces (by the multiplication map). Theorem 2.3 holds true for $H_I(k)$ with the role of $H_0(k)$ replaced by $H_{0,I}(k)$. We call $H_{0,I}(k)$ and $H_I(k)$ standard parabolic subalgebras of $H_0(k)$ and $H(k)$, respectively. Note that $H_0(k)$ (respectively $H(k)$) is a free right $H_{0,I}(k)$-module (respectively $H_I(k)$-module) with basis $\{T_w\}_{w \in W_0^I}$.

We write $H_I^a(k)$ for the unital complex subalgebra of $H^a(k)$ generated by $Y^\lambda$ ($\lambda \in Q^\vee$) and $T_i$ ($i \in I$). The following technical lemma will be convenient at a later stage.

**Lemma 2.4.** *The standard parabolic algebra $H_I^a(k)$ is algebraically generated by $T_i$ ($i \in I$) and $Y^{\pm w^{-1}(\varphi^\vee)}$ ($w \in \widetilde{W}_0^I$), where $\widetilde{W}_0^I$ is a complete set of representatives of the coset space $W_0/W_{0,I}$.*

*Proof.* It is clear that $H_I^a(k)$ is algebraically generated by $T_i$ ($i \in I$) and $Y^{\pm w^{-1}(\varphi^\vee)}$ ($w \in W_0$).

For $\alpha \in R_0$ we write $\sigma(\alpha) = 1$ if $\alpha \in R_0^+$ and $\sigma(\alpha) = -1$ if $\alpha \in R_0^-$. Then

$$Y^{w^{-1}(\varphi^\vee)} = T_w^{-1} T_0^{\sigma(w^{-1}\varphi)} T_{s_\varphi w} \tag{2.10}$$

and

$$T_{ws_i} = T_w T_i^{\sigma(w\alpha_i)} \tag{2.11}$$

in $H^a(k)$ for all $w \in W_0$ and $i \in \{1, \ldots, n\}$, see [23, (3.3.6)] and [23, (3.1.7)].

Let $i \in I$ and $w \in W_0$. If $\sigma(w^{-1}\varphi) \neq \sigma(s_i w^{-1}\varphi)$ then $w^{-1}\varphi = \alpha_i$ or $= -\alpha_i$, hence

$$Y^{s_i w^{-1}(\varphi^\vee)} = Y^{-w^{-1}(\varphi)}.$$

If $\sigma(w^{-1}\varphi)=\sigma(s_iw^{-1}\varphi)$ then

$$\begin{aligned}Y^{s_iw^{-1}(\varphi^\vee)}&=T_i^{-\sigma(w\alpha_i)}\big(T_w^{-1}T_0^{\sigma(w^{-1}\varphi)}T_{s_\varphi w}\big)T_i^{\sigma(s_\varphi w\alpha_i)}\\&=T_i^{-\sigma(w\alpha_i)}Y^{w^{-1}(\varphi^\vee)}T_i^{\sigma(s_\varphi w\alpha_i)}.\end{aligned}$$

This shows that the $Y^{\pm w^{-1}(\varphi^\vee)}$ with $w\in\widetilde{W}_0^I$, together with the $T_i$ $(i\in I)$, already form algebraic generators of $H_I^a(k)$. $\square$

2.4. **The affine Hecke algebra modules.** We have two characters (algebra maps) $\epsilon_\pm^k: H(k)\to\mathbb{C}$, characterized by $\epsilon_\pm^k(T_j)=\pm k_j^{\pm1}$ $(0\le j\le n)$ and $\epsilon_\pm^k(T_\omega)=1$ $(\omega\in\Omega)$. The character $\epsilon_+^k$ (respectively $\epsilon_-^k$) is called the trivial (respectively Steinberg) character of $H(k)$. From e.g. [23, §2.4],

$$\epsilon_\pm^k(T_{\tau(\lambda)})=\prod_{\alpha\in R_0^+}\big(\pm k_{\alpha^\vee}\big)^{\pm\langle\lambda,\alpha\rangle},\qquad \lambda\in P_+^\vee. \tag{2.12}$$

Set

$$\delta_\pm^k:=\prod_{\alpha\in R_0^+}\big(\pm k_{\alpha^\vee}\big)^{\pm\alpha}\in T, \tag{2.13}$$

i.e. it is the element of $T=\mathrm{Hom}_{\mathbb{Z}}(P^\vee,\mathbb{C}^\times)$ mapping the coweight $\lambda\in P^\vee$ to $\prod_{\alpha\in R_0^+}\big(\pm k_{\alpha^\vee}\big)^{\pm\langle\lambda,\alpha\rangle}$. Then it follows from (2.12) that

$$\epsilon_\pm^k(f(Y))=f\big(\delta_\pm^k\big),\qquad f\in\mathbb{C}[T]. \tag{2.14}$$

More generally, we will consider $H(k)$-modules induced from characters of a standard parabolic subalgebra $H_I(k)$. The characters will be parametrized by elements of

$$T_I^k:=\{\gamma\in T\mid \gamma^{\alpha_i^\vee}=k_i^2\quad\forall\, i\in I\}. \tag{2.15}$$

Note that $\delta_\pm^k\in T_{[1,n]}^{k^{\pm1}}$, where $[1,n]:=\{1,\ldots,n\}$ and $k^{-1}$ is the multiplicity function $R\ni a\mapsto k_a^{-1}$.

Observe that

$$w\gamma=\gamma\prod_{\alpha\in R_0^{I,+}\cap wR_0^{I,-}}k_{\alpha^\vee}^{-2\alpha}\qquad(\gamma\in T_I^k,\ w\in W_{0,I}), \tag{2.16}$$

hence in particular

$$\underline{w_0}\gamma=\rho_I^k\gamma\qquad(\gamma\in T_I^k) \tag{2.17}$$

where

$$\rho_I^k:=\prod_{\alpha\in R_0^{I,+}}k_{\alpha^\vee}^{-2\alpha}\in T. \tag{2.18}$$

**Lemma 2.5.** *Let $\gamma\in T_I^{k^{\pm1}}$.*
**(i)** *There exists a unique character $\chi_\gamma^{k,\pm,I}: H_I(k)\to\mathbb{C}$ satisfying $\chi_\gamma^{k,\pm,I}(f(Y))=f(\gamma)$ and $\chi_\gamma^{k,\pm,I}(T_i)=\pm k_i^{\pm1}$ for $f\in\mathbb{C}[T]$ and $i\in I$.*
**(ii)** *The left $H(k)$-module*

$$M^{k,\pm,I}(\gamma):=\mathrm{Ind}_{H_I(k)}^{H(k)}\big(\chi_\gamma^{k,\pm,I}\big)$$

*has complex linear basis*

$$v_w^{k,\pm,I}(\gamma):=T_w\otimes_{H_I(k),\chi_\gamma^{k,\pm,I}}1,\qquad w\in W_0^I.$$

*Proof.* It suffices to show that $\chi_\gamma^{k,\pm,I}$ preserves the cross relation (2.9) for $f\in\mathbb{C}[T]$ and $i\in I$. The cross relation is explicitly given by

$$f(Y)T_i=T_i(s_if)(Y)+(k_i^{-1}-k_i)\left(\frac{(s_if)(Y)-f(Y)}{1-Y^{-\alpha_i^\vee}}\right). \tag{2.19}$$

Fix $i\in I$ and $f\in\mathbb{C}[T]$. If $k_i^2=1$ then (2.19) reduces to $f(Y)T_i=T_i(s_if)(Y)$, which is indeed respected by $\chi_\gamma^{k,\pm,I}$ since $s_i\gamma=\gamma$ for $\gamma\in T_I^{k^{\pm1}}$ by (2.16). If $k_i^2\neq 1$, then (2.19) is respected by $\chi_\gamma^{k,\pm,I}$ since $\gamma^{\alpha_i^\vee}=k_i^{\pm2}$ for $\gamma\in T_I^{k^{\pm1}}$ and

$$\pm k_i^{\pm1}f(\gamma)=\pm k_i^{\pm1}f(s_i\gamma)+(k_i^{-1}-k_i)\left(\frac{f(s_i\gamma)-f(\gamma)}{1-k_i^{\mp2}}\right).$$

□

The modules $M^k(\gamma):=M^{k,\pm,\emptyset}(\gamma)$ ($\gamma\in T$) are called the principal series modules of $H(k)$. We write $v_w^k(\gamma):=v_w^{k,\pm,\emptyset}(\gamma)$ ($w\in W_0$) for the corresponding standard basis elements.

**Example 2.6.** *In view of* (2.14), $M^{k,\pm,[1,n]}(\delta_\pm^k)$ *is the one-dimensional* $H(k)$*-module characterized by the algebra map* $\epsilon_\pm^k: H(k)\to\mathbb{C}$.

By Lemma 2.1, the action of the finite Hecke algebra $H_0(k)$ on the standard basis $\{v_w^{k,\pm,I}(\gamma)\}_{w\in W_0^I}$ of $M^{k,\pm,I}(\gamma)$ is given by

$$T_iv_w^{k,\pm,I}(\gamma)=\begin{cases}(k_i-k_i^{-1})v_w^{k,\pm,I}(\gamma)+v_{s_iw}^{k,\pm,I}(\gamma), & \text{if } w\in A_i,\\ v_{s_iw}^{k,\pm,I}(\gamma), & \text{if } w\in B_i,\\ \pm k_i^{\pm1}v_w^{k,\pm,I}(\gamma), & \text{if } w\in C_i.\end{cases} \tag{2.20}$$

Furthermore, $f(Y)v_e^{k,\pm,I}(\gamma)=f(\gamma)v_e^{k,\pm,I}(\gamma)$ for all $f\in\mathbb{C}[T]$.

2.5. **Intertwiners.** By a well known result of Bernstein, the center $Z(H(k))$ of the affine Hecke algebra equals $\mathcal{A}_Y^{k,W_0}$ (the subalgebra of $H(k)$ consisting of elements $f(Y)$ with $f\in\mathbb{C}[T]^{W_0}$).

Let $M$ be a finite dimensional left $H(k)$-module and $\gamma\in T$. We write

$$M^{W_0\gamma}:=\{m\in M\mid f(Y)m=f(\gamma)m\quad\forall f\in\mathbb{C}[T]^{W_0}\},$$

which is a $H(k)$-submodule of $M$. We furthermore write

$$M_\gamma:=\{m\in M\mid f(Y)m=f(\gamma)m\quad\forall f\in\mathbb{C}[T]\}.$$

**Definition 2.7.** *Let $M$ be a finite dimensional left $H(k)$-module.*
**(i)** *$M$ is said to have central character $W_0\gamma\in T/W_0$ if $M=M^{W_0\gamma}$.*
**(ii)** *We say that $M$ is calibrated if $M=\bigoplus_{\gamma\in T}M_\gamma$.*

Note that the $H(k)$-module $M^{k,\pm,I}(\gamma)$ ($\gamma\in T_I^{k^{\pm1}}$) has central character $W_0\gamma$.

We now determine the conditions on $\gamma\in T_I^{k^{\pm1}}$ that ensure that $M^{k,\pm,I}(\gamma)$ is calibrated using the intertwiners of $H(k)$. In the following theorem we collect the definitions and the basic properties of the intertwiners (cf., e.g., [24, 18], [29, §2.2]).

**Theorem 2.8.** *For $1\le i\le n$ set*

$$I_i(k):=T_i(1-Y^{\alpha_i^\vee})+(k_i-k_i^{-1})Y^{\alpha_i^\vee}\in H(k).$$

*There exists unique elements $I_w(k)\in H(k)$ ($w\in W_0$) satisfying*

$$I_w(k)=I_{i_1}(k)I_{i_2}(k)\cdots I_{i_r}(k)$$

*if* $w = s_{i_1} s_{i_2} \cdots s_{i_r} \in W_0$ *is a reduced expression* ($1 \leq i_j \leq n$)*. Furthermore,*

$$I_i(k)^2 = (k_i - k_i^{-1} Y^{\alpha_i^\vee})(k_i - k_i^{-1} Y^{-\alpha_i^\vee})$$

*for* $1 \leq i \leq n$ *and in* $H(k)$*,*

$$I_w(k) f(Y) = (wf)(Y) I_w(k) \qquad (w \in W_0,\, f \in \mathbb{C}[T]).$$

Using (2.9), we have the alternative expression

$$I_i(k) = (1 - Y^{-\alpha_i^\vee}) T_i + k_i^{-1} - k_i, \qquad 1 \leq i \leq n \tag{2.21}$$

for the intertwiners of the affine Hecke algebra $H(k)$.

**Corollary 2.9.** *For* $w \in W_0$ *we have*

$$I_w(k) I_{w^{-1}}(k) = d_w(Y)(w d_{w^{-1}})(Y)$$

*in* $H(k)$*, with* $d_w \in \mathbb{C}[T]$ *given by*

$$d_w(t) = \prod_{\alpha \in R_0^+ \cap w R_0^-} (k_{\alpha^\vee} - k_{\alpha^\vee}^{-1} t^{\alpha^\vee}).$$

For $\gamma \in T_I^{k^{\pm 1}}$ and $w \in W_0^I$ we set

$$b_w^{k,\pm,I}(\gamma) := I_w(k) v_e^{k,\pm,I}(\gamma) \in M^{k,\pm,I}(\gamma)_{w\gamma}. \tag{2.22}$$

*Remark* 2.10. If $w \in W_0 \setminus W_0^I$ and $\gamma \in T_I^{k^{\pm 1}}$ then $I_w(k) v_e^{k,\pm,I}(\gamma) = 0$. To prove this it suffices to note that $I_i(k) v_e^{k,\pm,I}(\gamma) = 0$ for $i \in I$, which is immediate from the definition of $I_i(k)$ and the fact that $\gamma^{\alpha_i^\vee} = k_i^{\pm 2}$.

**Lemma 2.11.** *Let* $\gamma \in T_I^{k^{\pm 1}}$*. For* $w \in W_0^I$ *we have*

$$b_w^{k,\pm,I}(\gamma) = \Big( \prod_{\alpha \in (R_0^+ \setminus R_0^{I,+}) \cap w^{-1} R_0^-} (1 - \gamma^{\alpha^\vee}) \Big) v_w^{k,\pm,I}(\gamma) + \sum_{u \in W_0^I : u < w} a_u^\pm v_u^{k,\pm,I}(\gamma)$$

*for some* $a_u^\pm \in \mathbb{C}$*.*

*Proof.* By induction on $l(w)$ we have for $w \in W_0$,

$$I_w(k) = T_w \prod_{\alpha \in R_0^+ \cap w^{-1} R_0^-} (1 - Y^{\alpha^\vee}) + \sum_{u \in W_0 : u < w} T_u f_u(Y)$$

in $H(k)$, for some $f_u \in \mathbb{C}[T]$. Thus, for $w \in W_0^I$,

$$b_w^{k,\pm,I}(\gamma) = \Big( \prod_{\alpha \in R_0^+ \cap w^{-1} R_0^-} (1 - \gamma^{\alpha^\vee}) \Big) v_w^{k,\pm,I}(\gamma) + \sum_{u \in W_0 : u < w} f_u(\gamma) T_u v_e^{k,\pm,I}(\gamma).$$

Since $w(R_0^{I,+}) \subseteq R_0^+$ for $w \in W_0^I$, the product over $\alpha$ is in fact a product over the set $(R_0^+ \setminus R_0^{I,+}) \cap w^{-1}(R_0^-)$. Furthermore, if $u \in W_0$, $w \in W_0^I$ and $u < w$, then $\overline{u} \leq u = \overline{u}\underline{u} < w$ and

$$T_u v_e^{k,\pm,I}(\gamma) = T_{\overline{u}} T_{\underline{u}} v_e^{k,\pm,I}(\gamma) = \epsilon_\pm^k(T_{\underline{u}}) v_{\overline{u}}^{k,\pm,I}(\gamma).$$

This completes the proof. □

**Proposition 2.12.** *If* $\gamma \in T_I^{k^{\pm 1}}$ *satisfies* $\gamma^{\alpha^\vee} \neq 1$ *for all* $\alpha \in R_0^+ \setminus R_0^{I,+}$*, then*

- **(i)** $w\gamma = w'\gamma$ *for* $w, w' \in W_0^I$ *if and only if* $w = w'$*.*
- **(ii)** $M^{k,\pm,I}(\gamma)$ *is calibrated.*

**(iii)** $M^{k,\pm,I}(\gamma)=\bigoplus_{w\in W_0^I}M^{k,\pm,I}(\gamma)_{w\gamma}$ *and* $\mathrm{Dim}_{\mathbb{C}}\big(M^{k,\pm,I}(\gamma)_{w\gamma}\big)=1$ *for all* $w\in W_0^I$.

**(iv)** $M^{k,\pm,I}(\gamma)_{w\gamma}=\mathbb{C}b_w^{k,\pm,I}(\gamma)$ *for all* $w\in W_0^I$.

*Proof.* **(i)** The fixpoint subgroup

$$W_{0,\gamma}:=\{w\in W_0\mid w\gamma=\gamma\}$$

of $\gamma$ is generated by the reflections $s_\alpha$ ($\alpha\in R_0^+$) it contains (see [33]). For $\alpha\in R_0^+$ we have $s_\alpha\in W_{0,\gamma}$ if and only if $\gamma^{\alpha^\vee}=1$. By the assumption that $\gamma^{\alpha^\vee}\neq 1$ for $\alpha\in R_0^+\setminus R_0^{I,+}$, we conclude that $s_\alpha\in W_{0,\gamma}$ ($\alpha\in R_0^+$) implies that $\alpha\in R_0^{I,+}$. Hence $W_{0,\gamma}\subset W_{0,I}$. The result now follows immediately.

**(ii)** follows from **(iii)**.

**(iii)**&**(iv)** The previous lemma shows that $b_w^{k,\pm,I}(\gamma)\in M^{k,\pm,I}(\gamma)_{w\gamma}$ is nonzero for all $w\in W_0^I$. By a dimension count and **(i)** we get $M^{k,\pm,I}(\gamma)=\bigoplus_{w\in W_0^I}M^{k,\pm,I}(\gamma)_{w\gamma}$ and $M^{k,\pm,I}(\gamma)_{w\gamma}=\mathbb{C}b_w^{k,\pm,I}(\gamma)$ for all $w\in W_0^I$. □

**Lemma 2.13.** *Let* $\gamma\in T_I^{k^{\pm1}}$ *such that* $\gamma^{\alpha^\vee}\neq 1$ *for all* $\alpha\in R_0^+\setminus R_0^{I,+}$. *Let* $1\leq i\leq n$. *If* $w\in A_i$ *then*

$$\begin{aligned}T_ib_w^{k,\pm,I}(\gamma)=&\frac{(k_i-k_i^{-1}\gamma^{w^{-1}(\alpha_i)^\vee})(k_i-k_i^{-1}\gamma^{-w^{-1}(\alpha_i)^\vee})}{(1-\gamma^{w^{-1}(\alpha_i)^\vee})}b_{s_iw}^{k,\pm,I}(\gamma)\\&+\frac{(k_i^{-1}-k_i)\gamma^{w^{-1}(\alpha_i)^\vee}}{(1-\gamma^{w^{-1}(\alpha_i)^\vee})}b_w^{k,\pm,I}(\gamma).\end{aligned}\tag{2.23}$$

*If* $w\in B_i$ *then*

$$T_ib_w^{k,\pm,I}(\gamma)=\frac{1}{(1-\gamma^{w^{-1}(\alpha_i)^\vee})}b_{s_iw}^{k,\pm,I}(\gamma)+\frac{(k_i^{-1}-k_i)\gamma^{w^{-1}(\alpha_i)^\vee}}{(1-\gamma^{w^{-1}(\alpha_i)^\vee})}b_w^{k,\pm,I}(\gamma).\tag{2.24}$$

*If* $w\in C_i$ *then* $T_ib_w^{k,\pm,I}(\gamma)=\pm k_i^{\pm1}b_w^{k,\pm,I}(\gamma)$.

*Proof.* If $w\in A_i$ then $l(s_iw)=l(w)-1$ hence $w^{-1}(\alpha_i)\in R_0^-$. Furthermore, $w^{-1}(\alpha_i)\notin R_0^{I,-}$ since $w\in W_0^I$. Consequently $\gamma^{w^{-1}(\alpha_i)^\vee}\neq 1$, hence (2.23) makes sense.

Similarly, if $w\in B_i$ then $w^{-1}(\alpha_i)\in R_0^+$ since $l(s_iw)=l(w)+1$ and $w^{-1}(\alpha_i)\notin R_0^{I,+}$ since $s_iw\in W_0^I$. Consequently $\gamma^{w^{-1}(\alpha_i)^\vee}\neq 1$, and (2.24) makes sense.

By Theorem 2.8 we have for $w\in W_0^I$,

$$\begin{aligned}(1-\gamma^{w^{-1}(\alpha_i)^\vee})T_ib_w^{k,\pm,I}(\gamma)&=T_i(1-Y^{\alpha_i^\vee})I_w(k)v_e^{k,\pm,I}(\gamma)\\&=(I_i(k)I_w(k)+(k_i^{-1}-k_i)Y^{\alpha_i^\vee}I_w(k))v_e^{k,\pm,I}(\gamma).\end{aligned}\tag{2.25}$$

If $w\in A_i$ then $I_i(k)I_w(k)=I_i(k)^2I_{s_iw}(k)=d_{s_i}(Y)(s_id_{s_i})(Y)I_{s_iw}(k)$ and (2.23) follows from (2.25), Theorem 3.4, and the fact that $s_iw\in W_0^I$. If $w\in B_i$ then $I_i(k)I_w(k)=I_{s_iw}(k)$ and $s_iw\in W_0^I$, hence (2.24) follows from (2.25). It remains to show that $T_ib_w^{k,\pm,I}(\gamma)=\pm k_i^{\pm1}b_w^{k,\pm,I}(\gamma)$ if $w\in C_i$.

Fix $w\in C_i$. Then $w^{-1}(\alpha_i)=\alpha_{i_w}$ for a unique $i_w\in I$ (cf. Lemma 2.1). In particular, $\gamma^{w^{-1}(\alpha_i)^\vee}=\gamma^{\alpha_{i_w}^\vee}=k_{i_w}^{\pm2}=k_i^{\pm2}$.

Since $l(s_iw)=l(w)+1$ it follows from (2.25) that

$$(1-\gamma^{w^{-1}(\alpha_i)^\vee})T_ib_w^{k,\pm,I}(\gamma)=I_{s_iw}(k)v_e^{k,\pm,I}(\gamma)+(k_i^{-1}-k_i)\gamma^{w^{-1}(\alpha_i)^\vee}b_w^{k,\pm,I}(\gamma).$$

Hence, using $\gamma^{w^{-1}(\alpha_i)^\vee}=k_i^{\pm 2}$, $I_{s_iw}(k)=I_w(k)I_{i_w}(k)$ and $I_j(k)v_e^{k,\pm,I}(\gamma)=0$ for $j\in I$,

$$(1-k_i^{\pm 2})T_ib_w^{k,\pm,I}(\gamma)=(k_i^{-1}-k_i)k_i^{\pm 2}b_w^{k,\pm,I}(\gamma).$$

Consequently $T_ib_w^{k,\pm,I}(\gamma)=\pm k_i^{\pm 1}b_w^{k,\pm,I}(\gamma)$ if $k_i^2\neq 1$.

It remains to consider the case that $w\in C_i$ and that $k_i^2=1$. Then $f(Y)T_i=T_i(s_if)(Y)$ in $H(k)$ for all $f\in\mathbb{C}[T]$ by (2.19) and $s_{i_w}\gamma=\gamma$ since $\gamma\in T_I^{k^{\pm 1}}$, hence

$$f(Y)T_ib_w^{k,\pm,I}(\gamma)=T_if(s_iw\gamma)b_w^{k,\pm,I}(\gamma)=f(ws_{i_w}\gamma)T_ib_w^{k,\pm,I}(\gamma)=f(w\gamma)T_ib_w^{k,\pm,I}(\gamma)$$

for all $f\in\mathbb{C}[T]$. This shows that $T_ib_w^{k,\pm,I}(\gamma)=c_i^{\pm}b_w^{k,\pm,I}(\gamma)$ for some $c_i^{\pm}\in\mathbb{C}$. Note that

$$T_iv_w^{k,\pm,I}(\gamma)=T_{s_iw}v_e^{k,\pm,I}(\gamma)=T_wT_{i_w}v_e^{k,\pm,I}(\gamma)=\pm k_i^{\pm 1}v_w^{k,\pm,I}(\gamma),$$

hence by Lemma 2.11,

$$T_ib_w^{k,\pm,I}(\gamma)=\pm k_i^{\pm 1}\lambda_wv_w^{k,\pm,I}(\gamma)+\sum_{u\in W_0^I:u<w}a_u^{\pm}T_iv_u^{k,\pm,I}(\gamma)$$

for some $a_u^{\pm}\in\mathbb{C}$ and with $\lambda_w:=\prod_{\alpha\in R_0^+\setminus R_0^{I,+}\cap w^{-1}R_0^-}(1-\gamma^{\alpha^\vee})\neq 0$. Let $u\in W_0^I$ with $u<w$. Since $w\in C_i$, hence $s_iw\notin W_0^I$, we have $\overline{s_iu}\neq w$. Furthermore, $T_iv_u^{k,\pm,I}\in\mathrm{span}_{\mathbb{C}}\{v_u^{k,\pm,I}(\gamma),v_{\overline{s_iu}}^{k,\pm,I}(\gamma)\}$ by (2.20). Hence

$$T_ib_w^{k,\pm,I}(\gamma)=\pm k_i^{\pm 1}\lambda_wv_w^{k,\pm,I}(\gamma)+\sum_{u\in W_0^I\setminus\{w\}}a_u'^{\pm}v_u^{k,\pm,I}(\gamma)$$

for some $a_u'^{\pm}\in\mathbb{C}$. On the other hand,

$$\begin{aligned}T_ib_w^{k,\pm,I}(\gamma)&=c_i^{\pm}b_w^{k,\pm,I}(\gamma)\\&=c_i^{\pm}\lambda_wv_w^{k,\pm,I}(\gamma)+\sum_{u\in W_0^I:u<w}c_i^{\pm}a_u^{\pm}v_u^{k,\pm,I}(\gamma).\end{aligned}$$

Comparing the coefficient of $v_w^{k,\pm,I}(\gamma)$ in these two expressions of $T_ib_w^{k,\pm,I}(\gamma)$, we conclude that $c_i^{\pm}=\pm k_i^{\pm 1}$. Hence $T_ib_w^{k,\pm,I}(\gamma)=\pm k_i^{\pm 1}b_w^{k,\pm,I}(\gamma)$ for $w\in C_i$. $\square$

### 2.6. **(Anti)spherical vectors.**

**Definition 2.14.** *Let $M$ be a left $H(k)$-module. We call*

$$M^{\pm}:=\{m\in M\mid hm=\epsilon_{\pm}^k(h)m\quad\forall\, h\in H_0(k)\}$$

*the space of spherical (+), respectively antispherical (−), elements in $M$.*

We also write for $M^{I,\pm}$ for the space of vectors $m\in M$ satisfying $hm=\epsilon_{\pm}^k(h)m$ for all $h\in H_{0,I}(k)$, so that $M^{\pm}=M^{[1,n],\pm}$. Define

$$C_{\pm}^I(k):=\sum_{w\in W_0^I}\epsilon_{\pm}^k(T_w)T_w\in H_0(k)\tag{2.26}$$

and write $C_{\pm}(k)=C_{\pm}^{\emptyset}(k)$.

**Lemma 2.15.** *Let $M$ be a left $H(k)$-module. Then the action of $C_{\pm}^I(k)$ on $M^{I,\pm}$ defines a linear map*

$$C_{\pm}^I(k):M^{I,\pm}\to M^{\pm}.$$

*Proof.* This follows by a direct computation using Lemma 2.1 and (2.20). $\square$

*Remark* 2.16. If $M$ is a left $H(k)$-module and $m \in M^{I,\pm}$ then

$$C_\pm(k)m = P_I(k^{\pm 1})(C^I_\pm(k)m)$$

with

$$P_I(k) := \sum_{w\in W_{0,I}} \epsilon^k_+(T_w)^2 \tag{2.27}$$

the Poincaré polynomial (see [22]) of $W_{0,I}$.

Of special interest for the present paper is the case that $M = M^{k,\pm,I}(\gamma)$ with $\gamma \in T_I^{k^{\pm 1}}$. Then we have $v_e^{k,\pm,I}(\gamma) \in M^{k,\pm,I}(\gamma)^{I,\pm}$, hence we obtain a spherical (+), respectively antispherical (−), vector

$$\begin{aligned} 1_\gamma^{k,\pm,I} :=&C^I_\pm(k)v_e^{k,\pm,I}(\gamma)\\ =&\sum_{w\in W_0^I}\epsilon^k_\pm(T_w)v_w^{k,\pm,I}(\gamma)\in M^{k,\pm,I}(\gamma)^\pm. \end{aligned}\tag{2.28}$$

Clearly $1_\gamma^{k,\pm,I}\neq 0$.

**Lemma 2.17.** *For all $\gamma \in T_I^{k^{\pm 1}}$ we have $M^{k,\pm,I}(\gamma)^\pm = \mathbb{C}1_\gamma^{k,\pm,I}$.*

*Proof.* Let $m \in M^{k,\pm,I}(\gamma)^\pm$ and write $m = \sum_{w\in W_0^I} a_w v_w^{k,\pm,I}(\gamma)$ with $a_w \in \mathbb{C}$ ($w \in W_0^I$). Then $a_w = \pm k_i^{\pm 1} a_{s_i w}$ for $w \in A_i$ in view of Lemma 2.1 and (2.20). Let $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in W_0^I$ be a reduced expression. Then $s_{i_j}s_{i_{j+1}}\cdots s_{i_r} \in A_{i_j}$ for $1\leq j\leq r$. Consequently $a_w = a_e\epsilon^k_\pm(T_w)$ for all $w\in W_0^I$. Thus $m = a_e 1_\gamma^{k,\pm,I}$. □

Recall from Proposition 2.12 that $\{b_w^{k,\pm,I}(\gamma)\}_{w\in W_0^I}$ is a complex linear basis of $M^{k,\pm,I}(\gamma)$ if $\gamma\in T_I^{k^{\pm1}}$ satisfies $\gamma^{\alpha^\vee}\neq 1$ for all $\alpha\in R_0^+\setminus R_0^{I,+}$. The special case $I=\emptyset$ of the following theorem goes back to Kato, see [18, Prop. 1.20] (in the p-adic group case, i.e. for constant multiplicity function $k$, see Casselman [1]).

**Theorem 2.18.** *Suppose that $\gamma\in T_I^{k^{\pm1}}$ and $\gamma^{\alpha^\vee}\neq 1$ for all $\alpha\in R_0^+\setminus R_0^{I,+}$. Let $M$ be a left $H(k)$-module and $m\in M$ satisfying $hm = \chi_\gamma^{k,\pm,I}(h)m$ for all $h\in H_I(k)$. Then*

$$\begin{aligned} C^I_\pm(k)m =& \Big(\prod_{\alpha\in R_0^+\setminus R_0^{I,+}}\frac{\pm k_{\alpha^\vee}^{\pm1}}{1-\gamma^{\alpha^\vee}}\Big)\\ &\times\sum_{w\in W_0^I}\left(\prod_{\alpha\in(R_0^+\setminus R_0^{I,+})\cap w^{-1}(R_0^+)}(\pm k_{\alpha^\vee}^{\mp1}\mp k_{\alpha^\vee}^{\pm1}\gamma^{\alpha^\vee})\right)I_w(k)m. \end{aligned}\tag{2.29}$$

*In particular, by taking $M = M^{k,\pm,I}(\gamma)$ and $m = v_e^{k,\pm,I}(\gamma)$,*

$$\begin{aligned} 1_\gamma^{k,\pm,I} =& \Big(\prod_{\alpha\in R_0^+\setminus R_0^{I,+}}\frac{\pm k_{\alpha^\vee}^{\pm1}}{1-\gamma^{\alpha^\vee}}\Big)\\ &\times\sum_{w\in W_0^I}\left(\prod_{\alpha\in(R_0^+\setminus R_0^{I,+})\cap w^{-1}(R_0^+)}(\pm k_{\alpha^\vee}^{\mp1}\mp k_{\alpha^\vee}^{\pm1}\gamma^{\alpha^\vee})\right)b_w^{k,\pm,I}(\gamma) \end{aligned}\tag{2.30}$$

*in $M^{k,\pm,I}(\gamma)$.*

*Proof.* If $m \in M$ satisfies $hm = \chi_\gamma^{k,\pm,I}(h)m$ for all $h \in H_I(k)$ then there exists a unique $H(k)$-linear map $M^{k,\pm,I}(\gamma) \to M$ mapping $v_e^{k,\pm,I}(\gamma)$ onto $m$. Hence it suffices to prove (2.30).

Let $\gamma \in T_I^{k^{\pm 1}}$ satisfying $\gamma^{\alpha^\vee} \neq 1$ for all $\alpha \in R_0^+ \setminus R_0^{I,+}$. Then we have a unique expansion $1_\gamma^{k,\pm,I} = \sum_{w \in W_0^I} a_w^\pm b_w^{k,\pm,I}(\gamma)$ in $M^{k,\pm,I}(\gamma)$ for some $a_w^\pm \in \mathbb{C}$. The fact that $T_i 1_\gamma^{k,\pm,I} = \pm k_i^{\pm 1} 1_\gamma^{k,I}$ for $1 \leq i \leq n$ implies, in view of the previous lemma and Lemma 2.1, the following recursion relation for the $a_w^\pm$:

$$\frac{(k_i^{-1} - k_i)\gamma^{w^{-1}(\alpha_i)^\vee}}{(1 - \gamma^{w^{-1}(\alpha_i)^\vee})} a_w^\pm + \frac{1}{(1 - \gamma^{-w^{-1}(\alpha_i)^\vee})} a_{s_i w}^\pm = \pm k_i^{\pm 1} a_w^\pm$$

if $1 \leq i \leq n$ and $w \in A_i$. Rewriting the recursion relation gives

$$a_{s_i w}^\pm = (\pm k_i^{\mp 1} \mp k_i^{\pm 1} \gamma^{-w^{-1}(\alpha_i)^\vee}) a_w^\pm$$

if $1 \leq i \leq n$ and $w \in A_i$.

Let $w \in W_0^I$. By Lemma 2.2 there exist $1 \leq i_j \leq n$ such that $\overline{w_0} = s_{i_1} s_{i_2} \cdots s_{i_r} w$ and $l(\overline{w_0}) = l(w) + r$. Then for $1 \leq j \leq r$,

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_r} w = s_{i_{j-1}} s_{i_{j-2}} \cdots s_{i_1} \overline{w_0} \in A_{i_j}.$$

Hence

$$\begin{aligned} a_w^\pm &= a_{s_{i_r}(s_{i_r} w)}^\pm \\ &= a_{s_{i_r} w}^\pm (\pm k_{i_r}^{\mp 1} \mp k_{i_r}^{\pm 1} \gamma^{w^{-1}(\alpha_{i_r})^\vee}) \\ &= \cdots = a_{\overline{w_0}}^\pm \prod_{\alpha \in (R_0^+ \setminus R_0^{I,+}) \cap w^{-1}(R_0^+)} (\pm k_{\alpha^\vee}^{\mp 1} \mp k_{\alpha^\vee}^{\pm 1} \gamma^{\alpha^\vee}), \end{aligned}$$

where we have used that

$$\begin{aligned} w(R_0^+ \setminus R_0^{I,+}) \cap R_0^+ &= w(R_0^+ \setminus R_0^{I,+} \cup R_0^{I,-}) \cap R_0^+ \\ &= w\overline{w_0}^{-1}(R_0^-) \cap R_0^+ \\ &= (\overline{w_0} w^{-1})^{-1} R_0^- \cap R_0^+ \\ &= \{\alpha_{i_r}, s_{i_r}(\alpha_{i_{r-1}}), \ldots, s_{i_r} s_{i_{r-1}} \cdots s_{i_2}(\alpha_{i_1})\}. \end{aligned}$$

It thus remains to show that

$$a_{\overline{w_0}}^\pm = \prod_{\alpha \in R_0^+ \setminus R_0^{I,+}} \frac{\pm k_{\alpha^\vee}^{\pm 1}}{1 - \gamma^{\alpha^\vee}} = \epsilon_\pm^k(T_{\overline{w_0}}) \prod_{\alpha \in R_0^+ \setminus R_0^{I,+}} \frac{1}{1 - \gamma^{\alpha^\vee}}. \tag{2.31}$$

By Lemma 2.11 and the fact that $R_0^+ \cap \overline{w_0}^{-1}(R_0^-) = R_0^+ \setminus R_0^{I,+}$, we have

$$\sum_{w \in W_0^I} a_w^\pm b_w^{k,\pm,I}(\gamma) = a_{\overline{w_0}}^\pm \Big( \prod_{\alpha \in R_0^+ \setminus R_0^{I,+}} (1 - \gamma^{\alpha^\vee}) \Big) v_{\overline{w_0}}^{k,\pm,I}(\gamma) + \sum_{u < \overline{w_0}} c_u^\pm v_u^{k,\pm,I}(\gamma)$$

for certain $c_u^\pm \in \mathbb{C}$. On the other hand, by definition this is equal to $1_\gamma^{k,\pm,I} = \sum_{w \in W_0^I} \epsilon_\pm^k(T_w) v_w^{k,\pm,I}(\gamma)$. Comparing the coefficient of $v_{\overline{w_0}}^{k,\pm,I}(\gamma)$ in both expressions gives (2.31). $\square$

## 3. Double affine Hecke algebras and quantum affine KZ equations

3.1. **The double affine Hecke algebra.** In this subsection we recall the definition of Cherednik's double affine Hecke algebra and we state some of its fundamentel properties. To keep the technicalities to a minimum, we only discuss the *twisted case* (in the sense that it is naturally associated to reduced twisted affine Lie algebras).

Let $m$ be the positive integer such that $m\langle P^\vee, P^\vee\rangle = \mathbb{Z}$. In the remainder of the paper we fix arbitrary multiplicity function $k$ and arbitrary $q^{\frac{1}{m}} \in \mathbb{C}^\times$ (it is allowed to be a root of unity), unless explicitly specified otherwise. We write

$$q^r := \big(q^{\frac{1}{m}}\big)^{mr}, \qquad r \in \frac{1}{m}\mathbb{Z}.$$

For $\lambda \in P^\vee$ define torus elements $q^\lambda \in T = \mathrm{Hom}_{\mathbb{Z}}\big(P^\vee, \mathbb{C}^\times\big)$ by $P^\vee \ni \mu \mapsto q^{\langle \lambda, \mu\rangle}$. We extend the $W_0$-action on $T$ to a $q^{\frac{1}{m}}$-dependent action of the extended affine Weyl group $W \simeq W_0 \ltimes P^\vee$ on $T$ by $\tau(\lambda)_q t = q^\lambda t$ for $\lambda \in P^\vee$ and $t \in T$.

The corresponding contragredient action of $W$ on $\mathcal{M}(T)$ is explicitly given by

$$\begin{aligned} (w_q f)(t) &= f(w^{-1}t), \qquad && w \in W_0,\\ (\tau(\lambda)_q f)(t) &= f(q^{-\lambda}t), \qquad && \lambda \in P^\vee. \end{aligned} \tag{3.1}$$

In particular, on the monomial basis $e^\mu$ ($\mu \in P^\vee$) of $\mathbb{C}[T]$ the $W$-action takes on the form

$$\begin{aligned} w_q(e^\mu) &= e^{w\mu}, \qquad && w \in W_0,\\ \tau(\lambda)_q(e^\mu) &= q^{-\langle \lambda,\mu\rangle}e^\mu, \qquad && \lambda \in P^\vee. \end{aligned}$$

We extend the definition of the monomials $e^\mu$ ($\mu \in P^\vee$) in such a way that the latter formulas can be captured in terms of a $W$-action on the exponents of the generalized monomials. Consider the $W$-invariant subset

$$\widehat{P}^\vee := \frac{1}{m}\mathbb{Z}c + P^\vee$$

of $V$. Note that $\widehat{P}^\vee$ contains the affine root system $R$. We set for $rc + \lambda \in \widehat{P}^\vee$ ($r \in \frac{1}{m}\mathbb{Z}$, $\lambda \in P^\vee$),

$$e_q^{rc+\lambda} := q^r e^\lambda \in \mathbb{C}[T].$$

Then it is an easy verification that

$$w_q\big(e_q^{\hat\mu}\big) = e_q^{w\hat\mu} \qquad (w \in W,\ \hat\mu \in \widehat{P}^\vee).$$

We write $t_q^{\hat\mu}$ for the evaluation of $e_q^{\hat\mu}$ at $t \in T$. We write $\mathbb{C}(T)\#_q W$ for the associated smashed product algebra (note that it depends on the choice $q^{\frac{1}{m}}$ of the $m$th root of $q$). It thus is $\mathbb{C}(T)\otimes_{\mathbb{C}}\mathbb{C}[W]$ as complex vector space, with the canonical embeddings of $\mathbb{C}(T)$ and $\mathbb{C}[W]$ algebra maps, and with cross relations governed by (3.1): $w\cdot p = (w_q p)\cdot w$ ($w \in W$, $p \in \mathbb{C}(T)$).

Note that $\mathbb{C}(T)\#_q W$ acts canonically on $\mathbb{C}(T)$ as $q$-difference reflection operators with coefficients from $\mathbb{C}(T)$. This action is faithful unless $q^{\frac{1}{m}}$ is a root of unity. Despite this fact, it is convenient to think of $\mathbb{C}(T)\#_q W$ as the algebra of $q$-difference reflection operators with coefficients from $\mathbb{C}(T)$.

Define $c_a^{k,q} \in \mathbb{C}(T)$ by

$$c_a^{k,q} := \frac{k_a^{-1} - k_a e_q^a}{1 - e_q^a} \qquad (a \in R). \tag{3.2}$$

It coincides with the definition (2.8) of $c_a^k$ when $a \in R_0^\vee$. We denote $c_j^{k,q} = c_{a_j}^{k,q}$ for $0 \leq j \leq n$. The following result is essentially due to Cherednik. The only difference is that we allow $q^{\frac{1}{m}}$ to be a root of unity.

**Theorem 3.1.** *There exists a unique injective unital algebra homomorphism $\pi^{k,q} : H(k) \to \mathbb{C}(T)\#_q W$ satisfying*

$$\begin{aligned} \pi^{k,q}(T_j) &= k_j + c_j^{k,q}(s_j - 1), \\ \pi^{k,q}(T_\omega) &= \omega \end{aligned}$$

*for $0 \leq j \leq n$ and $\omega \in \Omega$.*

*Proof.* The (by now standard) arguments showing that the above formulas give rise to a unique algebra homomorphism $\pi^{k,q} : H(k) \to \mathbb{C}(T)\#_q W$ are valid without restrictions on $k$ and $q^{\frac{1}{m}}$ (see, e.g., [23, §4.3]). It remains to show that $\pi^{k,q}$ is injective. This follows from a simple modification of the proof of [23, (4.3.11)], working in $\mathbb{C}(T)\#_q W$ instead of in $\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(T)\big)$ (the latter being the image space for the representation map associated to the canonical action of $\mathbb{C}(T)\#_q W$ on $\mathbb{C}(T)$). □

The following result is due to Cherednik [6, Theorem 2.1].

**Theorem 3.2.** *Up to isomorphism, there exists a unique complex, unital, associative algebra $\mathbb{H}(k,q)$ satisfying the following properties.*

- **(i)** *$\mathbb{C}[T]$ and $H(k)$ are subalgebras of $\mathbb{H}(k)$,*
- **(ii)** *the multiplication map defines a linear isomorphism $\mathbb{C}[T] \otimes_{\mathbb{C}} H(k) \xrightarrow{\sim} \mathbb{H}(k,q)$,*
- **(iii)** *for all $f \in \mathbb{C}[T]$, $0 \leq j \leq n$ and $\omega \in \Omega$ we have in $\mathbb{H}(k,q)$,*

$$\begin{aligned} T_j f &= (s_{j,q} f) T_j + (c_j^{k,q} - k_j)((s_{j,q} f) - f), \\ \omega f &= (\omega_q f)\omega. \end{aligned} \tag{3.3}$$

*Proof.* The modification of the proof of [23, (4.3.11)] (see the proof of the previous theorem) shows that $\{e^\lambda \pi^{k,q}(T_w)\}_{\lambda \in P^\vee, w \in W}$ is $\mathbb{C}$-linear independent in $\mathbb{C}(T)\#_q W$ (also if $q^{\frac{1}{m}}$ is a root of unity). Consequently $\mathbb{H}(k,q)$ can be realized as the subalgebra of $\mathbb{C}(T)\#_q W$ generated by $\mathbb{C}[T]$ and $\pi^{k,q}(H(k))$. □

The algebra $\mathbb{H}(k,q)$ is called the double affine Hecke algebra (note that it depends on the choice $q^{\frac{1}{m}}$ of the $m$th root of $q$).

Since $\mathbb{C}[T]^\times := \mathbb{C}[T] \setminus \{0\} \subset \mathbb{H}(k,q)$ is a left Ore set, we can form the corresponding left localized double affine Hecke algebra $\mathbb{H}_{loc}(k,q)$. Theorem 3.2 is valid for $\mathbb{H}_{loc}(k,q)$ with the role of $\mathbb{C}[T]$ replaced by $\mathbb{C}(T)$; we will call it the localized version of Theorem 3.2. By (the proof of) Theorem 3.2, the algebra homomorphism $\pi^{k,q} : H(k) \to \mathbb{C}(T)\#_q W$ uniquely extends to an injective algebra homomorphism

$$\mathbb{H}_{loc}(k,q) \to \mathbb{C}(T)\#_q W$$

mapping $f \in \mathbb{C}(T) \subset \mathbb{H}_{loc}(k,q)$ to $f$ viewed as element in $\mathbb{C}(T)\#_q W$. The resulting algebra homomorphism will again be denoted by $\pi^{k,q} : \mathbb{H}_{loc}(k,q) \to \mathbb{C}(T)\#_q W$.

*Remark* 3.3. Composing $\pi^{k,q}$ with the algebra map $\mathbb{C}(T)\#_q W \to \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(T)\big)$ arising from the canonical action of $\mathbb{C}(T)\#_q W$ on $\mathbb{C}(T)$ as $q$-difference reflection operators, turns $\mathbb{C}(T)$ into a left $\mathbb{H}_{loc}(k,q)$-module. Restricting the action to the double affine Hecke algebra $\mathbb{H}(k,q)$, the algebra $\mathbb{C}[T]$ of regular functions on $T$

becomes a $\mathbb{H}(k,q)$-invariant subspace of $\mathbb{C}(T)$. The resulting left $\mathbb{H}(k,q)$-module $\mathbb{C}[T]$ is Cherednik's basic representation of $\mathbb{H}(k,q)$. It is faithful unless $q^{\frac{1}{m}}$ is a root of unity.

Observe that $\pi^{k,q}:\mathbb{H}_{loc}(k,q)\rightarrow\mathbb{C}(T)\#_qW$ is in fact an algebra isomorphism. The pre-images of the $s_j\in\mathbb{C}(T)\#_qW$ are given by

$$(\pi^{k,q})^{-1}(s_j)=\big(c_j^{k,q}\big)^{-1}\big(T_j-k_j+c_j^{k,q}\big)\in\mathbb{H}_{loc}(k,q),\qquad 0\leq j\leq n. \tag{3.4}$$

They are called the normalized ($X$-)intertwiners of the localized double affine Hecke algebra.

### 3.2. **Algebras of $H(k)$-valued $q$-difference reflection operators.**

**Definition 3.4.** *Denote $\mathbb{C}_\sigma^q[T]$ (respectively $\mathbb{C}_\nabla^{k,q}[T]$) for the subalgebra of $\mathbb{C}(T)$ generated by $\mathbb{C}[T]$ and $(1-e_q^a)^{-1}$ for all $a\in R$ (respectively $\mathbb{C}[T]$ and $(k_a^{-1}-k_ae_q^a)^{-1}$ for all $a\in R$).*

Note that $\mathbb{C}_\sigma^q[T]$ (respectively $\mathbb{C}_\nabla^{k,q}[T]$) is a $W$-module subalgebra of $\mathbb{C}(T)$ with respect to the action (3.1), and it contains $c_a^{k,q}$ (respectively $(c_a^{k,q})^{-1}$) for all $a\in R$. The possible singularities of $f\in\mathbb{C}_\sigma^q[T]$, respectively $f\in\mathbb{C}_\nabla^{k,q}[T]$, are at

$$\mathcal{S}_\sigma^q:=\{t\in T\mid t_q^a=1\ \text{ for some } a\in R\},$$

respectively at

$$\mathcal{S}_\nabla^{k,q}:=\{t\in T\mid t_q^a=k_a^2\ \text{ for some } a\in R\}.$$

Note that $\mathcal{S}_\nabla^{k,q}=\mathcal{S}_\nabla^{k^{-1},q}$ and that $T_I^k\subseteq\mathcal{S}_\nabla^{k,q}$ if $I\neq\emptyset$.

We can now form the smashed product algebras $\mathbb{C}_\sigma^q[T]\#_qW$ and $\mathbb{C}_\nabla^{k,q}[T]\#_qW$, which are subalgebras of $\mathbb{C}(T)\#_qW$. The algebras of $H(k)$-valued $q$-difference reflection algebras with coefficients from $\mathbb{C}_\sigma^q[T]$, $\mathbb{C}_\nabla^{k,q}[T]$ and $\mathbb{C}(T)$ are

$$\begin{aligned}\mathcal{A}_\sigma^{k,q}&:=\mathbb{C}_\sigma^q[T]\#_qW\otimes_{\mathbb{C}}H(k),\\ \mathcal{A}_\nabla^{k,q}&:=\mathbb{C}_\nabla^{k,q}[T]\#_qW\otimes_{\mathbb{C}}H(k).\end{aligned}\tag{3.5}$$

and

$$\mathcal{A}^{k,q}:=\mathbb{C}(T)\#_qW\otimes_{\mathbb{C}}H(k),$$

respectively. We will identify the algebras $\mathbb{C}(T)\#_qW$ and $H(k)$ with their canonical images in $\mathcal{A}^{k,q}$ (and similarly in case of $\mathcal{A}_\sigma^{k,q}$ and $\mathcal{A}_\nabla^{k,q}$). The following statement is essentially a reformulation of [4, Theorem 2.3].

**Proposition 3.5.** *There exists a unique algebra homomorphism $\sigma^{k,q}:\mathbb{H}_{loc}(k,q)\rightarrow\mathcal{A}^{k,q}$ satisfying*

$$\begin{aligned}\sigma^{k,q}(f)&=f,\\ \sigma^{k,q}(T_j)&=s_jT_j+(c_j^{k,q}-k_j)(s_j-1),\\ \sigma^{k,q}(T_\omega)&=\omega T_\omega\end{aligned}\tag{3.6}$$

*for $f\in\mathbb{C}(T)$, $0\leq j\leq n$ and $\omega\in\Omega$. Furthermore, $\sigma^{k,q}\big(\mathbb{H}(k,q)\big)\subseteq\mathcal{A}_\sigma^{k,q}$.*

*Proof.* A direct verification shows that the assignments (3.6) respect the cross relations (3.3), as well as the relations $T_\omega T_jT_{\omega^{-1}}=T_{\omega(j)}$ in $H(k)\subset\mathbb{H}(k,q)$ ($\omega\in\Omega$ and $0\leq j\leq n$). It thus remains to show that the $\sigma^{q,k}(T_j)\in\mathcal{A}^{k,q}$ ($0\leq j\leq n$) from

(3.6) satisfy the defining relations (2.5) of $H^a(k)$, for which it suffices to provide a proof if $q^{\frac{1}{m}}$ is not a root of unity.

Suppose that $q^{\frac{1}{m}}$ is not a root of unity. Consider the $\mathbb{H}_{loc}(k,q)$-module

$$\mathrm{Ind}_{H(k)}^{\mathbb{H}_{loc}(k,q)}\big(H(k)\big),$$

where $H(k)$ is considered as left $H(k)$-module by left multiplication. By **(i)** of the localized version of Theorem 3.2 it is isomorphic to $\mathbb{C}(T)\otimes_{\mathbb{C}} H(k)$ as a complex vector space. Denote the resulting representation map by

$$\sigma : \mathbb{H}_{loc}(k,q) \to \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(T)\otimes_{\mathbb{C}} H(k)\big).$$

Since $q^{\frac{1}{m}}$ is not a root of unity, the formula

$$(pw\otimes h) : r\otimes h' \mapsto p(w_q r)\otimes hh'$$

for $p, r \in \mathbb{C}(T)$, $w\in W$ and $h, h' \in H(k)$ defines an algebra embedding

$$\mathcal{A}^{k,q} \hookrightarrow \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(T)\otimes_{\mathbb{C}} H(k)\big).$$

We identify $\mathcal{A}^{k,q}$ with its image in $\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(T)\otimes_{\mathbb{C}} H(k)\big)$. By a direct computation using **(ii)** and **(iii)** of the localized version of Theorem 3.2, it follows that $\sigma(f)$, $\sigma(T_j)$ and $\sigma(T_\omega)$ for $f\in\mathbb{C}(T)$, $0\le j\le n$ and $\omega\in\Omega$ are given by (3.6). In particular, they lie in the subalgebra $\mathcal{A}^{k,q}$. Thus $\sigma$ is an algebra homomorphism $\sigma:\mathbb{H}_{loc}(k,q)\to\mathcal{A}^{k,q}$, satisfying (3.6).

The last statement of the proposition is immediate. □

Note that $\pi^{k,q} = \big(\mathrm{id}\otimes\epsilon_+^k\big)\circ\sigma^{k,q}$, where $\pi^{k,q} : \mathbb{H}_{loc}(k,q)\to\mathbb{C}(T)\#_qW$ is the algebra isomorphism as defined in the previous subsection. In particular, $\pi^{k,q}\big(\mathbb{H}(k,q)\big)\subseteq\mathbb{C}_\sigma^q[T]\#_qW$.

*Remark* 3.6. Let $-k^{-1}$ be the multiplicity function on $R$ that takes the value $-k_a^{-1}$ on $a\in R$. Since

$$c_a^{k,q}(t)-k_a = c_a^{-k^{-1},q}(t)+k_a^{-1}\quad(a\in R),$$

we have a unique algebra isomorphism ${}^\dagger : \mathbb{H}_{loc}(k,q)\xrightarrow{\sim}\mathbb{H}_{loc}(-k^{-1},q)$ satisfying $T_j^\dagger = T_j$ $(0\le j\le n)$, $T_\omega^\dagger = T_\omega$ $(\omega\in\Omega)$ and $f^\dagger = f$ $(f\in\mathbb{C}(T))$. Then $\pi^{-k^{-1},q}\circ{}^\dagger = (\mathrm{id}\otimes\epsilon_-^k)\circ\sigma^{k,q}$.

**Corollary 3.7.** *Fix a multiplicity function $k$ on $R$ and fix $q^{\frac{1}{m}}\in\mathbb{C}^\times$. There exists a unique algebra homomorphism $\nabla^{k,q}:\mathbb{C}(T)\#_qW\to\mathcal{A}^{k,q}$ satisfying*

$$\begin{aligned}
&\nabla^{k,q}(f) = f,\\
&\nabla^{k,q}(s_j) = (c_j^{k,q})^{-1}s_jT_j + \frac{c_j^{k,q}-k_j}{c_j^{k,q}}s_j,\\
&\nabla^{k,q}(\omega) = \omega T_\omega
\end{aligned}\tag{3.7}$$

*for $f\in\mathbb{C}(T)$, $0\le j\le n$ and $\omega\in\Omega$. Furthermore, $\nabla^{k,q}(\mathbb{C}[T]\#_qW)\subseteq\mathcal{A}_\nabla^{k,q}$.*

*Proof.* Consider the algebra homomorphism

$$\nabla^{k,q} := \sigma^{k,q}\circ\big(\pi^{k,q}\big)^{-1} : \mathbb{C}(T)\#_qW\to\mathcal{A}^{k,q}.$$

A direct computation using (3.4) shows that $\nabla^{k,q}$ satisfies (3.7). The second statement is immediate. □

The algebra homomorphism $\nabla^{k,q}$ is the key ingredient in the definition of Cherednik's [4] quantum affine Knizhnik-Zamolodhikov (KZ) equations. We discuss this in detail in subsection 3.4.

3.3. **Characterizations of spaces of invariants.** For a complex, unital associative algebra $A$ we denote by $\mathrm{Mod}_A$ the category of left $A$-modules.

Proposition 3.5 gives rise to a covariant functor $F_\sigma : \mathrm{Mod}_{\mathcal{A}_\sigma^{k,q}} \to \mathrm{Mod}_{H(k)}$ in the following way. If $M$ is a left $\mathcal{A}_\sigma^{k,q}$-module $M$, then $F_\sigma(M)$ is the vector space $M$ with $H(k)$-module structure defined by

$$h \cdot m := \sigma^{k,q}(h)m \qquad (h \in H(k) \subset \mathbb{H}(k,q),\ \ m \in M).$$

Similarly, Corollary 3.7 gives rise to a covariant functor $F_\nabla : \mathrm{Mod}_{\mathcal{A}_\nabla^{k,q}} \to \mathrm{Mod}_{\mathbb{C}[W]}$. In this case $F_\nabla(M)$, for a left $\mathcal{A}_\nabla^{k,q}$-module $M$, is the vector space $M$ with $\mathbb{C}[W]$-module structure given by

$$w \cdot m := \nabla^{k,q}(w)m \qquad (w \in W,\ \ m \in M).$$

*Remark* 3.8. Since $\mathbb{C}_\sigma^q[T]\#_q W$ and $H(k)$ are mutually commuting subalgebras of $\mathcal{A}_\sigma^{k,q}$, both $\pi^{k^{-1},q}(H(k^{-1})) \subseteq \mathbb{C}_\sigma^q[T]\#_q W$ and $H(k)$ act on a left $\mathcal{A}_{\sigma,\nabla}^{k,q}$-module $M$, and these actions commute. It is important to carefully distinguish between these two commuting actions.

The next aim is to relate certain invariant subspaces of $F_\sigma(M)$ and of $F_\nabla(M)$ in case $M$ is a left module over the subalgebra $\mathcal{A}_{\sigma,\nabla}^{k,q}$ of $\mathcal{A}^{k,q}$, generated by $\mathcal{A}_\sigma^{k,q}$ and $\mathcal{A}_\nabla^{k,q}$. We first need to introduce some more notations.

Write $\mathbb{C}_{\sigma,\nabla}^{k,q}[T] \subseteq \mathbb{C}(T)$ for the subalgebra generated by $\mathbb{C}[T]$, $(1-e_q^a)^{-1}$ and $(k_a^{-1} - k_a e_q^a)^{-1}$ for all $a \in R$. Note that $\mathbb{C}_{\sigma,\nabla}^{k,q}[T] = \mathbb{C}_{\sigma,\nabla}^{k^{-1},q}[T]$. Then

$$\mathbb{C}[T] \subseteq \mathbb{C}_\sigma^q[T], \mathbb{C}_\nabla^{k,q}[T] \subseteq \mathbb{C}_{\sigma,\nabla}^{k,q}[T] \subseteq \mathbb{C}(T)$$

as $W$-module algebras. The elements $c_a^{k,q}$ $(a \in R)$ are invertible in $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]$. Let $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$ be the algebra of $q$-difference reflection operators with coefficients in $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]$. Then

$$\mathcal{A}_{\sigma,\nabla}^{k,q} = \mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W \otimes_{\mathbb{C}} H(k)$$

and $\mathcal{A}_\sigma^{k,q}, \mathcal{A}_\nabla^{k,q} \subseteq \mathcal{A}_{\sigma,\nabla}^{k,q} \subseteq \mathcal{A}^{k,q}$ as algebras.

**Definition 3.9.** *For a left $\mathcal{A}_{\sigma,\nabla}^{k,q}$-module $M$ we write $M_\sigma = F_\sigma\big(M|_{\mathcal{A}_\sigma^{k,q}}\big)$ and $M_\nabla = F_\nabla\big(M|_{\mathcal{A}_\nabla^{k,q}}\big)$ for the associated $H(k)$-module and $\mathbb{C}[W]$-module, respectively.*

For a left $\mathcal{A}_{\sigma,\nabla}^{k,q}$-module $M$, for a subalgebra $A \subseteq H(k)$ and for a subgroup $G \subseteq W$ we now write

$$\begin{aligned} M_\sigma^A &:= \{m \in M \mid \sigma^{k,q}(a)m = \epsilon_+^k(a)m \quad \forall\, a \in A\}, \\ M_\nabla^G &:= \{m \in M \mid \nabla^{k,q}(g)m = m \quad \forall\, g \in G\}. \end{aligned}$$

Let $J_k : H(k^{-1}) \to H(k)$ be the unique unital algebra anti-involution satisfying $J_k(T_j) = T_j^{-1}$ $(0 \leq j \leq n)$ and $J_k(T_\omega) = T_{\omega^{-1}}$ $(\omega \in \Omega)$. Note that $J_k$ restricts to an algebra anti-involution $J_k : H_0(k^{-1}) \to H_0(k)$.

**Proposition 3.10.** *Let $M$ be a left $\mathcal{A}^{k,q}_{\sigma,\nabla}$-module.*
**(i)** *We have*

$$M_\sigma^{H_0(k)} = M_\nabla^{W_0} = \{m \in M \mid \pi^{k^{-1},q}(h)m = J_k(h)m \quad \forall\, h \in H_0(k^{-1})\}. \tag{3.8}$$

**(ii)** *We have*

$$M_\sigma^{H(k)} = M_\nabla^{W} = \{m \in M \mid \pi^{k^{-1},q}(h)m = J_k(h)m \quad \forall\, h \in H(k^{-1})\}. \tag{3.9}$$

*Proof.* Observe that in $\mathcal{A}^{k,q}_{\sigma,\nabla}$,

$$\begin{aligned} \sigma^{k,q}(T_j) &= k_j + c_j^{k,q}(\nabla^{k,q}(s_j) - 1), \\ \sigma^{k,q}(T_\omega) &= \nabla^{k,q}(\omega) \end{aligned}$$

for $0 \leq j \leq n$ and $\omega \in \Omega$. Since $c_j^{k,q}$ is invertible in $\mathcal{A}^{k,q}_{\sigma,\nabla}$, it implies for $m \in M$,

$$\begin{aligned} \sigma^{k,q}(T_j)m = k_j m \;\Leftrightarrow\; \nabla^{k,q}(s_j)m = m, \\ \sigma^{k,q}(T_\omega)m = m \;\Leftrightarrow\; \nabla^{k,q}(\omega)m = m \end{aligned}$$

for $0 \leq j \leq n$ and $\omega \in \Omega$. This implies the first equalities in (3.8) and (3.9). For the second equalities in (3.8) and (3.9) it suffices to show, for $m \in M$,

$$\begin{aligned} \nabla^{k,q}(s_j)m = m \;\Leftrightarrow\; \pi^{k^{-1},q}(T_j)m = J_k(T_j)m, \\ \nabla^{k,q}(\omega)m = m \;\Leftrightarrow\; \pi^{k^{-1},q}(T_\omega)m = J_k(T_\omega)m \end{aligned}$$

for $0 \leq j \leq n$ and $\omega \in \Omega$. The second equivalence is immediate. It thus remains to prove the first equivalence.

Note that $\nabla^{k,q}(s_j)m = m$ is equivalent to

$$s_j T_j m + (c_j^{k,q} - k_j)s_j m = c_j^{k,q} m.$$

We now act by $s_j \in \mathcal{A}^{k,q}_{\sigma,\nabla}$ on both sides, and pull the action of $s_j$ to the right. Using the fact that

$$w_q\big(c_a^{k,q}\big) = c_{wa}^{k,q}, \qquad c_{-a}^{k,q} = c_a^{k^{-1},q}$$

in $\mathbb{C}(T)$ for $w \in W$ and $a \in R$, it follows that $\nabla^{k,q}(s_j)m = m$ is equivalent to

$$T_j m + (c_j^{k^{-1},q} - k_j)m = c_j^{k^{-1},q} s_j m.$$

Since $T_j^{-1} = T_j - k_j + k_j^{-1}$ in $H(k)$, we conclude that $\nabla^{k,q}(s_j)m = m$ if and only if

$$T_j^{-1} m = (k_j^{-1} + c_j^{k^{-1},q}(s_j - 1))m.$$

The left hand side equals $J_k(T_j)m$ and the right hand side equals $\pi^{k^{-1},q}(T_j)m$, hence the result. □

*Remark* 3.11. The third form of the space of invariants (the far right side of (3.9)) is used in the analysis of special solutions of quantum Knizhnik-Zamolodchikov equations in the context of the Razumov-Stroganov conjectures, see, e.g., [30, §4.1].

3.4. **Quantum affine Knizhnik-Zamolodchikov equations.** In this subsection we recall Cherednik's [4] construction of the quantum affine KZ equations.

Observe that for $w \in W$,

$$\nabla^{k,q}(w) = C_w^{k,q} w \in \mathcal{A}_\nabla^{k,q}, \tag{3.10}$$

with $C_w^{k,q}$ an element in the subalgebra $\mathbb{C}_\nabla^{k,q}[T] \otimes_{\mathbb{C}} H(k)$ of $\mathcal{A}_\nabla^{k,q}$. It follows that the $C_w^{k,q}$ satisfies the cocycle conditions

$$C_{ww'}^{k,q} = C_w^{k,q} w_q\big(C_{w'}^{k,q}\big), \qquad w, w' \in W, \tag{3.11}$$

where $w_q$ acts on the first tensor component of $C_{w'}^{k,q} \in \mathbb{C}_\nabla^{k,q}[T] \otimes H(k)$. In view of the cocycle condition (3.11), the $C_w^{k,q}$ are uniquely determined by $C_{s_j}^{k,q}$ $(0 \leq j \leq n)$ and $C_\omega^{k,q}$ $(\omega \in \Omega)$. By (3.7), for $0 \leq j \leq n$ and $\omega \in \Omega$,

$$\begin{aligned} C_{s_j}^{k,q}(t) &= c_j^{k,q}(t)^{-1} T_j + \frac{c_j^{k,q}(t) - k_j}{c_j^{k,q}(t)} = \frac{T_j^{-1} - t_q^{a_j} T_j}{k_j^{-1} - k_j t_q^{a_j}}, \\ C_\omega^{k,q}(t) &= T_\omega \end{aligned} \tag{3.12}$$

as rational $H(k)$-valued functions in $t \in T$.

For a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module $L$ and a left $H(k)$-module $L$ we write

$$\Gamma_L^{k,q}(N) := L \otimes_{\mathbb{C}} N$$

for the associated $\mathcal{A}_{\sigma,\nabla}^{k,q}$-module. Typically, $L$ is some field of functions on $T$ (for example, $L = \mathbb{C}(T)$, or $\mathcal{M}(T)$) with $\mathbb{C}_{\sigma,\nabla}^{q,k}[T]\#_q W$ acting by $q$-difference reflection operators, in which case it is convenient to think of the resulting $\mathcal{A}_{\sigma,\nabla}^{k,q}$-module $\Gamma_L^{k,q}(N)$ as some space of global sections of a trivial vector bundle over $T$ with fiber $N$. We call $\Gamma_L^{k,q}(N)$ the space of $N$-valued functions on $T$ of class $L$.

The action of $\tau(P^\vee)$ on $\Gamma_L^{k,q}(N)_\nabla$ then gets the interpretation of an integrable $q$-connection on $\Gamma_L^{k,q}(N)$; in this interpretation the cocycle values $C_{\tau(\lambda)}^{k,q}$ $(\lambda \in P^\vee)$, acting on $\Gamma_L^{k,q}(N)$, play the role of the $q$-connection matrices, while the integrability is captured by the cocycle condition (3.11). The $W_0$-submodule $\Gamma_L^{k,q}(N)_\nabla^{\tau(P^\vee)}$ of $\Gamma_L^{k,q}(N)_\nabla$ then plays the role of the subspace of flat $q$-sections.

**Definition 3.12.** *The system $\nabla^{k,q}(\tau(P^\vee))$ of holonomic $q$-difference equations on $\Gamma_L^{k,q}(N)$ is the quantum affine Knizhnik-Zamolodchikov equations for $N$-valued functions on $T$ of class $L$.*

Note that the assignment

$$L \times N \to \Gamma_L^{k,q}(N)_\nabla \tag{3.13}$$

defines a covariant functor $\mathrm{Mod}_{\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W} \times \mathrm{Mod}_{H(k)} \to \mathrm{Mod}_{\mathbb{C}[W]}$. In particular, if $L \to L'$ and $N \to N'$ are morphisms of $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$ and $H(k)$-modules respectively, it gives rise to canonical linear maps

$$\Gamma_L^{k,q}(N)_\nabla^{\tau(P^\vee)} \to \Gamma_{L'}^{k,q}(N')_\nabla^{\tau(P^\vee)}, \qquad \Gamma_L^{k,q}(N)_\nabla^W \to \Gamma_{L'}^{k,q}(N')_\nabla^W.$$

## 4. Spectral problem of the Cherednik-Dunkl operators

For a left $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW$-module $L$ and a left $H(k)$-module of the form $M^{k,\pm,I}(\gamma)$ ($\gamma\in T_I^{k^{\pm1}}$) we relate in this section the space $\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W$ of $W_0$-invariant flat $q$-sections of the quantum affine KZ equations to a suitable space of common eigenfunctions of the Cherednik-Dunkl $q$-difference reflection operators $\pi^{k^{-1},q}(f(Y))\in\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW$ ($f\in\mathbb{C}[T]$) acting on $L$.

4.1. **$W_0$-invariants.** In this subsection we analyze $\Gamma_L^{k,q}(M^{k,\pm,I}(\gamma))_\nabla^{W_0}$ ($\gamma\in T_I^{k^{\pm1}}$). In the following subsection we extend the analysis to its subspace $\Gamma_L^{k,q}(M^{k,\pm,I}(\gamma))_\nabla^W$ of $W_0$-invariant flat $q$-sections.

If $L$ is a left $\mathbb{C}[T]^{k,q}_{\sigma,\nabla}[T]\#_qW$-module, then we write

$$L_\pi^{I,\pm}:=\{\phi\in L\mid \pi^{k^{-1},q}(h)\phi=\epsilon_\pm^{k^{-1}}(h)\phi\quad\forall\, h\in H_{0,I}(k^{-1})\}. \tag{4.1}$$

We give first an alternative description of the spaces $L_\pi^{I,\pm}$. Set $\rho^\vee:=\frac{1}{2}\sum_{\alpha\in R_0^+}\alpha^\vee\in P^\vee$ and define $G^{k,\pm}\in\mathbb{C}[T]$ by

$$G^{k,\epsilon}(t):=\begin{cases}1 & \text{if } \epsilon=+,\\ t^{\rho^\vee}\prod_{\alpha\in R_0^+}\big(k_{\alpha^\vee}^{-1}-k_{\alpha^\vee}t^{-\alpha^\vee}\big), & \text{if } \epsilon=-.\end{cases}$$

Note that $G^{k,\pm}\in\mathbb{C}^{k,q}_{\sigma,\nabla}[T]^\times$. Define for $w\in W_0$,

$$w_\pm:=\frac{G^{k,\pm}}{w(G^{k,\pm})}w\in\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW. \tag{4.2}$$

It is just $w$ in the symmetric case ($+$), but it is convenient to write it as $w_+$ to maintain an uniform treatment of the symmetric and antisymmetric theory.

**Lemma 4.1.** *Let $L$ be a left $\mathbb{C}[T]^{k,q}_{\sigma,\nabla}\#_qW$-module. Then $L_\pi^{I,\pm}=L^{W_{0,I},\pm}$, where*

$$L^{W_{0,I},\pm}:=\{\phi\in L\mid w_\pm\phi=\phi\quad\forall\, w\in W_{0,I}\}.$$

*Proof.* For all $i\in\{1,\ldots,n\}$ and $\phi\in L$ we have

$$(\pi^{k^{-1},q}(T_i)-k_i^{-1})\phi=c_i^{k^{-1},q}(s_i-1)\phi,$$
$$(\pi^{k^{-1},q}(T_i)+k_i)\phi=\left(\frac{k_i^{-1}-k_it^{\alpha_i^\vee}}{1-t^{\alpha_i^\vee}}\right)\left(1-\frac{G^{k,-}}{s_i(G^{k,-})}s_i\right)\phi$$

The lemma follows now immediately. □

**Proposition 4.2.** *Fix $\gamma\in T_I^{k^{\pm1}}$. Let $L$ be a left $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW$-module and let $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)$. Then $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W_0}$ if and only if*

$$\psi=\sum_{w\in W_0^I}\pi^{k^{-1},q}(T_{w^{-1}}^{-1})\phi\otimes v_w^{k,\pm,I}(\gamma)$$

*for some $\phi\in L_\pi^{I,\pm}$.*

*Proof.* Any $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)$ has a unique expansion of the form

$$\psi=\sum_{w\in W_0^I}\psi_w\otimes v_w^{k,\pm,I}(\gamma)$$

with $\psi_w \in L$. By (3.8) the condition $\psi \in \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W_0}$ is equivalent to

$$\sum_{w\in W_0^I} \pi^{k^{-1},q}(T_i)\psi_w \otimes v_w^{k,\pm,I}(\gamma) = \sum_{w\in W_0^I} \psi_w \otimes T_i^{-1} v_w^{k,\pm,I}(\gamma) \tag{4.3}$$

for all $i \in \{1,\ldots,n\}$, in view of (3.8). Recasting (4.3) as explicit recursion relations for the $\psi_w$ using Lemma 2.1 and the explicit formulas (2.20) for the action of the $T_i$ on the standard basis of $M^{k,\pm,I}(\gamma)$, implies that (4.3) holds for $1 \le i \le n$ if and only if

$$\pi^{k^{-1},q}(T_i)\psi_w = \begin{cases} \psi_{s_iw} & \text{if } \ w \in A_i, \\ \psi_{s_iw} + (k_i^{-1} - k_i)\psi_w & \text{if } \ w \in B_i, \\ \pm k_i^{\mp 1}\psi_w & \text{if } \ w \in C_i \end{cases} \tag{4.4}$$

for $1 \le i \le n$.

Suppose that we have a solution $\{\psi_w\}_{w\in W_0^I} \subset L$ of the recursion relations (4.4) for $1 \le i \le n$. Consider first the recursion (4.4) for $w = e \in W_0^I$ the unit element of $W_0$ and for $i \in I$. Then $e \in C_i$, $\overline{s_ie} = e$ and $i_e = i$, hence $\pi^{k^{-1},q}(T_i)\psi_e = \pm k_i^{\mp1}\psi_e$. It follows that $\psi_e \in L_\pi^{I,\pm}$. If $w \in W_0^I$ with $l(w) > 0$ and $w = s_{i_1}s_{i_2}\cdots s_{i_{l(w)}}$ is a reduced expression ($1 \le i_j \le n$), then repeated application of the first recursion in (4.4) shows that

$$\psi_w = \pi^{k^{-1},q}(T_{i_1}^{-1})\psi_{s_{i_2}\cdots s_{i_{l(w)}}} = \cdots = \pi^{k^{-1},q}(T_{w^{-1}}^{-1})\psi_e.$$

Hence a solution $\{\psi_w\}_{w\in W_0^I} \subset L$ of the recursion relations (4.4) for $1 \le i \le n$ is uniquely determined by $\psi_e$, and $\psi_e$ must be an element from the subspace $L_\pi^{I,\pm}$ of $L$.

On the other hand, let $\phi \in L_\pi^{I,\pm}$ and define

$$\psi_w := \pi^{k^{-1},q}(T_{w^{-1}}^{-1})\phi \in L, \qquad w \in W_0^I.$$

Then $\{\psi_w\}_{w\in W_0^I}$ satisfies the recursion relations (4.4) for $1 \le i \le n$ due to the following identities in $H_0(k^{-1})$,

$$T_iT_{w^{-1}}^{-1} = \begin{cases} T_{(s_iw)^{-1}}^{-1} & \text{if } \ w \in A_i, \\ T_{(s_iw)^{-1}}^{-1} + (k_i^{-1} - k_i)T_{w^{-1}}^{-1} & \text{if } \ w \in B_i, \\ T_{w^{-1}}^{-1}T_{i_w}^{-1} + (k_i^{-1} - k_i)T_{w^{-1}}^{-1} & \text{if } \ w \in C_i \end{cases}$$

for $1 \le i \le n$. The verification of these identities is straightforward. □

**Corollary 4.3.** *Let $\gamma \in T_I^{k^{\pm1}}$. We have a complex linear isomorphism*

$$L_\pi^{I,\pm} \xrightarrow{\sim} \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W_0},$$

*defined by*

$$\phi \mapsto \sum_{w\in W_0^I} \pi^{k^{-1},q}\big(T_{w^{-1}}^{-1}\big)\phi \otimes v_w^{k,\pm,I}(\gamma). \tag{4.5}$$

4.2. $W$**-invariants.** For the analysis of the $W$-invariants in $\Gamma_L^{k,q}(M^{k,\pm,I}(\gamma))_\nabla$ it is convenient to reformulate Corollary 4.3 in the following way.

For $i\in I$ let $i_I^*\in\{1,\ldots,n\}$ such that $\overline{w_0}(\alpha_i)=\alpha_{i_I^*}$, and set $I^*=\{i_I^*\}_{i\in I}$. It follows from the identities

$$T_{\overline{w_0}^{-1}}^{-1}T_iT_{\overline{w_0}^{-1}}=T_{i_I^*},\qquad \forall\, i\in I$$

in $H_0(k^{-1})$ that $L_\pi^{I^*,\pm}=\pi^{k^{-1},q}\big(T_{\overline{w_0}^{-1}}^{-1}\big)L_\pi^{I,\pm}$. Thus Corollary 4.3 can be reformulated as follows.

**Corollary 4.4.** *Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module and $\gamma\in T_I^{k^{\pm1}}$. We have a linear isomorphism*

$$L_\pi^{I^*,\pm}\overset{\sim}{\longrightarrow}\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W_0},$$

*defined by*

$$\phi\mapsto\psi_\phi:=\sum_{w\in W_0^I}\pi^{k^{-1},q}\big(T_{w\overline{w_0}^{-1}}\big)\phi\otimes v_w^{k,\pm,I}(\gamma). \tag{4.6}$$

*Proof.* It follows from Corollary 4.3 that we have a linear isomorphism

$$L_\pi^{I^*,\pm}\overset{\sim}{\longrightarrow}\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W_0}$$

given by

$$\phi\mapsto\sum_{w\in W_0^I}\pi^{k^{-1},q}\big(T_{w^{-1}}^{-1}T_{\overline{w_0}^{-1}}\big)\phi\otimes v_w^{k,\pm,I}(\gamma).$$

It thus suffices to show that $T_{w^{-1}}^{-1}T_{\overline{w_0}^{-1}}=T_{w\overline{w_0}^{-1}}$ in $H_0(k^{-1})$ if $w\in W_0^I$. This follows from the fact that for $w\in W_0^I$,

$$\begin{aligned}l(w\overline{w_0}^{-1})&=l(w\underline{w_0}w_0)=l(w_0)-l(w\underline{w_0})\\&=l(w_0)-l(\underline{w_0})-l(w)=l(\overline{w_0}^{-1})-l(w).\end{aligned}$$

□

The following lemma should be compared to Lemma 2.5**(i)**.

**Lemma 4.5.** *If $\gamma\in T_I^k$ then $\overline{w_0}\gamma^{-1}\in T_{I^*}^{k^{-1}}$. In particular, for $\gamma\in T_I^{k^{\pm1}}$ we have a well defined character $\chi_{\overline{w_0}\gamma^{-1}}^{k^{-1},\pm,I^*}:H_{I^*}(k^{-1})\to\mathbb{C}$.*

*Proof.* Let $\gamma\in T_I^k$. Since $\overline{w_0}(\alpha_i)=\alpha_{i_I^*}$ $(i\in I)$ we have, for $i\in I$,

$$(\overline{w_0}\gamma^{-1})^{\alpha_{i_I^*}^\vee}=\gamma^{-\alpha_i^\vee}=k_i^{-2}=k_{i_I^*}^{-2}.$$

Hence $\overline{w_0}\gamma^{-1}\in T_{I^*}^{k^{-1}}$.

The second statement follows from Lemma 2.5**(i)**. □

**Definition 4.6.** *Let $\gamma\in T_I^{k^{\pm1}}$. For a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module $L$ we define*

$$\begin{aligned}L_{\pi,a}^{I^*,\pm}[\overline{w_0}\gamma^{-1}]&:=\{\phi\in L\mid \pi^{k^{-1},q}(h)\phi=\chi_{\overline{w_0}\gamma^{-1}}^{k^{-1},\pm,I^*}(h)\phi\quad\forall h\in H_{I^*}^a(k^{-1})\},\\ L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]&:=\{\phi\in L\mid \pi^{k^{-1},q}(h)\phi=\chi_{\overline{w_0}\gamma^{-1}}^{k^{-1},\pm,I^*}(h)\phi\quad\forall h\in H_{I^*}(k^{-1})\}.\end{aligned}$$

Note the alternative description

$$L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]=\{\phi\in L_\pi^{I^*,\pm}\mid \pi^{k^{-1},q}(f(Y))\phi=f(\overline{w_0}\gamma^{-1})\phi\quad \forall f\in\mathbb{C}[T]\},$$

which emphasizes the fact that $L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$ consists of common eigenfunctions within $L_\pi^{I^*,\pm}$ of the commuting Cherednik-Dunkl operators $\pi^{k^{-1},q}(f(Y))$ ($f\in\mathbb{C}[T]$), with associated spectral point $\overline{w_0}\gamma^{-1}$. It is convenient to make explicit contact with the notations of Subsection 2.5. We write $L_\pi$ for a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module $L$ when we view $L$ as a $H(k^{-1})$-module via the algebra map $\pi^{k^{-1},q}:H(k^{-1})\to\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$. Then we can alternatively write for $\gamma\in T_I^{k^{\pm1}}$,

$$L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]=L_\pi^{I^*,\pm}\cap L_{\pi,\overline{w_0}\gamma^{-1}},$$

where, recall, $L_{\pi,\overline{w_0}\gamma^{-1}}$ is the $\mathcal{A}_Y^{k^{-1}}$-weight space of the $H(k^{-1})$-module $L_\pi$ with weight $\overline{w_0}\gamma^{-1}\in T$,

$$L_{\pi,\overline{w_0}\gamma^{-1}}=\{\phi\in L\mid \pi^{k^{-1},q}(f(Y))\phi=f(\overline{w_0}\gamma^{-1})\phi\quad\forall\, f\in\mathbb{C}[T]\}.$$

In particular, for $I^*=\emptyset$ we have

$$L_\pi^{\emptyset,\pm}[\overline{w_0}\gamma^{-1}]=L_{\pi,\overline{w_0}\gamma^{-1}}.$$

Similar alternative descriptions can be given of the space $L_{\pi,a}^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$.

**Proposition 4.7.** *Let $\gamma\in T_I^{k^{\pm1}}$. Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module. The map $\phi\mapsto\psi_\phi$, given by* (4.6)*, restricts to an isomorphism*

$$L_{\pi,a}^{I^*,\pm}[\overline{w_0}\gamma^{-1}]\xrightarrow{\sim}\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W^a}.$$

*Proof.* Let $\gamma\in T_I^{k^{\pm1}}$ and $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W_0}$, written as

$$\psi=\sum_{w\in W_0^I}\psi_w\otimes v_w^{k,\pm,I}(\gamma),$$

with $\psi_w=\pi^{k^{-1},q}\big(T_{w\overline{w}_0^{-1}}\big)\phi$ and $\phi\in L_\pi^{I^*,\pm}$, cf. Corollary 4.4. Then we have $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W^a}$ if and only if

$$\sum_{w\in W_0^I}\pi^{k^{-1},q}(T_0)\psi_w\otimes v_w^{k,\pm,I}(\gamma)=\sum_{w\in W_0^I}\psi_w\otimes T_0^{-1}v_w^{k,\pm,I}(\gamma)$$

in view of Proposition 3.10. For $w\in W_0^I$ we write $s_\varphi w=\overline{s_\varphi w}w_\varphi$ with, as usual, $\overline{s_\varphi w}\in W_0^I$ the minimal coset representative of $s_\varphi wW_{0,I}$, and $w_\varphi=\underline{s_\varphi w}\in W_{0,I}$. By (2.10) and the definition of $v_w^{k,\pm,I}(\gamma)$ we have for $w\in W_0^I$,

$$T_0^{-1}v_w^{k,\pm,I}(\gamma)=\begin{cases}\epsilon_\pm^k(T_{w_\varphi})\gamma^{-w^{-1}(\varphi^\vee)}v_{\overline{s_\varphi w}}^{k,\pm,I}(\gamma) & \text{if } w^{-1}\varphi\in R_0^+,\\ \epsilon_\pm^k(T_{w_\varphi})\gamma^{-w^{-1}(\varphi^\vee)}v_{\overline{s_\varphi w}}^{k,\pm,I}(\gamma)+(k_0^{-1}-k_0)v_w^{k,\pm,I}(\gamma) & \text{if } w^{-1}\varphi\in R_0^-.\end{cases}$$

Note that $w\mapsto\overline{s_\varphi w}$ defines an involution $W_0^I\xrightarrow{\sim}W_0^I$, and that $(\overline{s_\varphi w})_\varphi=w_\varphi^{-1}$ for all $w\in W_0^I$. It follows that $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W^a}$ if and only if

$$\begin{aligned}\sum_{w\in W_0^I}\pi^{k^{-1},q}(T_0)\psi_w\otimes v_w^{k,\pm,I}(\gamma)=&\sum_{w\in W_0^I}\epsilon_\pm^k(T_{w_\varphi})\gamma^{w_\varphi w^{-1}(\varphi^\vee)}\psi_{\overline{s_\varphi w}}\otimes v_w^{k,\pm,I}(\gamma)\\&+(k_0^{-1}-k_0)\sum_{w\in W_0^I:w^{-1}\varphi\in R_0^-}\psi_w\otimes v_w^{k,\pm,I}(\gamma),\end{aligned}$$

which holds true if and only if

$$\pi^{k^{-1},q}\big(T_0^{\sigma(w^{-1}\varphi)}\big)\psi_w=\epsilon_\pm^k(T_{w_\varphi})\gamma^{w_\varphi w^{-1}(\varphi^\vee)}\psi_{\overline{s_\varphi w}},\qquad \forall\, w\in W_0^I.$$

By (2.10) and using $T_{u^{-1}}^{-1}T_{w_0}=T_{uw_0}$ for $u\in W_0$,

$$\begin{aligned}T_0^{\sigma(w^{-1}\varphi)}T_{w^{-1}}^{-1}T_{\overline{w_0}^{-1}}&=T_0^{\sigma((s_\varphi ww_0)^{-1}\varphi)}T_{ww_0}T_{w_0}^{-1}T_{\overline{w_0}^{-1}}\\&=T_{w^{-1}s_\varphi}^{-1}T_{w_0}Y^{-w_0w^{-1}(\varphi^\vee)}T_{w_0}^{-1}T_{\overline{w_0}^{-1}}\\&=T_{(\overline{s_\varphi w})^{-1}}^{-1}T_{w_\varphi^{-1}}^{-1}T_{\underline{w_0}}T_{\overline{w_0}^{-1}}Y^{-w_0w^{-1}(\varphi^\vee)}T_{\overline{w_0}^{-1}}^{-1}T_{\underline{w_0}}^{-1}T_{\overline{w_0}^{-1}}.\end{aligned}$$

Since $\pi^{k^{-1},q}(T_{\overline{w_0}^{-1}})\phi\in L_\pi^{I,\pm}$, we obtain for $w\in W_0^I$,

$$\begin{aligned}\pi^{k^{-1},q}\big(T_0^{\sigma(w^{-1}\varphi)}\big)\psi_w&=\pi^{k^{-1},q}\big(T_0^{\sigma(w^{-1}\varphi)}T_{w^{-1}}^{-1}T_{\overline{w_0}^{-1}}\big)\phi\\&=\epsilon_\pm^k(T_{\underline{w_0}})\pi^{k^{-1},q}\big(T_{(\overline{s_\varphi w})^{-1}}^{-1}T_{w_\varphi^{-1}}^{-1}T_{\underline{w_0}}T_{\overline{w_0}^{-1}}Y^{-w_0w^{-1}(\varphi^\vee)}\big)\phi,\end{aligned}$$

while

$$\epsilon_\pm^k(T_{w_\varphi})\gamma^{w_\varphi w^{-1}(\varphi^\vee)}\psi_{\overline{s_\varphi w}}=\epsilon_\pm^k(T_{w_\varphi})\gamma^{w_\varphi w^{-1}(\varphi^\vee)}\pi^{k^{-1},q}\big(T_{(\overline{s_\varphi w})^{-1}}^{-1}T_{\overline{w_0}^{-1}}\big)\phi.$$

Hence $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W^a}$ if and only if, for all $w\in W_0^I$,

$$\pi^{k^{-1},q}\big(Y^{-w_0w^{-1}(\varphi^\vee)}\big)\phi=\gamma^{w_\varphi w^{-1}(\varphi^\vee)}\phi.$$

Here we have used repeatedly that $\pi^{k^{-1},q}(T_{\overline{w_0}^{-1}})\phi\in L_\pi^{I,\pm}$.

Note that $\{ww_0\}_{w\in W_0^I}$ is a complete set of coset representatives of $W_0/W_{0,I^*}$. In view of Lemma 2.4 it thus remains to show that for $\gamma\in T_I^k$,

$$\gamma_I^{-w_0w^{-1}(\varphi^\vee)}=\gamma^{w_\varphi w^{-1}(\varphi^\vee)}\qquad (w\in W_0^I),$$

with $\gamma_I\in T_{I^*}^{k^{-1}}$ given by

$$\gamma_I:=\overline{w_0}\gamma^{-1}=w_0(\rho_I^k\gamma)^{-1}\tag{4.7}$$

(see Lemma 4.5), where the last equality follows from (2.17). Hence it remains to show for $\gamma\in T_I^k$,

$$(\rho_I^k)^{w^{-1}(\varphi^\vee)}\gamma^{w^{-1}(\varphi^\vee)}=\gamma^{w_\varphi w^{-1}(\varphi^\vee)}\qquad (w\in W_0^I).$$

Let $w\in W_0^I$. Since $\gamma\in T_I^k$ and $w_\varphi\in W_{0,I}$ we have

$$w_\varphi^{-1}\gamma=\gamma\prod_{\alpha\in R_0^{I,+}\cap w_\varphi^{-1}R_0^{I,-}}k_\alpha^{-2\alpha}$$

in $T$ by (2.16), hence it suffices to show that for all $w\in W_0^I$,

$$(\rho_I^k)^{w^{-1}(\varphi^\vee)}=\prod_{\alpha\in R_0^{I,+}\cap w_\varphi^{-1}R_0^{I,-}}k_{\alpha^\vee}^{-2\langle\alpha,w^{-1}\varphi^\vee\rangle},$$

i.e. that $\langle\alpha,w^{-1}\varphi^\vee\rangle=0$ if $\alpha\in R_0^{I,+}\cap w_\varphi^{-1}R_0^{I,+}$ and $w\in W_0^I$. This follows from Lemma 4.8 below. □

**Lemma 4.8.** *Let $w\in W_0^I$ and write $s_\varphi w=\overline{s_\varphi w}w_\varphi$ with $\overline{s_\varphi w}\in W_0^I$ and $w_\varphi=\underline{s_\varphi w}\in W_{0,I}$. Then*

$$\langle w\alpha,\varphi^\vee\rangle=\begin{cases}1 & \text{if }\ \alpha\in R_0^{I,+}\cap w_\varphi^{-1}R_0^{I,-},\\ 0 & \text{if }\ \alpha\in R_0^{I,+}\cap w_\varphi^{-1}R_0^{I,+}.\end{cases}$$

*Proof.* Let $\alpha \in R_0^{I,+}$. Then $w\alpha \in R_0^+$ since $w \in W_0^I$. Using

$$s_\varphi(w\alpha) = w\alpha - \langle w\alpha, \varphi^\vee \rangle \varphi$$

and using the fact that $\varphi \in R_0^+$ is the longest root, we conclude that $\langle w\alpha, \varphi^\vee\rangle$ is 0 or 1. It is 0 if $s_\varphi(w\alpha) \in R_0^+$ and 1 if $s_\varphi(w\alpha) \in R_0^-$. Furthermore,

$$R_0^+ \cap (s_\varphi w)^{-1} R_0^- = w_\varphi^{-1}\big(R_0^+ \cap (\overline{s_\varphi w})^{-1} R_0^-\big) \cup \big(R_0^{I,+} \cap w_\varphi^{-1} R_0^{I,-}\big)$$

since $l(\overline{s_\varphi w} w_\varphi) = l(\overline{s_\varphi w}) + l(w_\varphi)$ and $w_\varphi \in W_{0,I}$.

Suppose that $\alpha \in R_0^{I,+} \cap w_\varphi^{-1} R_0^{I,-}$. Then $\alpha \in R_0^+ \cap (s_\varphi w)^{-1} R_0^-$, hence $s_\varphi(w\alpha) \in R_0^-$. Consequently, $\langle w\alpha, \varphi^\vee\rangle = 1$.

Suppose that $\alpha \in R_0^{I,+} \cap w_\varphi^{-1} R_0^{I,+}$. Then

$$s_\varphi(w\alpha) = \overline{s_\varphi w} w_\varphi(\alpha) \in \overline{s_\varphi w}\big(R_0^{I,+}\big) \subseteq R_0^+,$$

hence $\langle w\alpha, \varphi^\vee\rangle = 0$. □

The following theorem provides an explicit bridge between the theory of quantum affine KZ equations and the Cherednik-Macdonald theory.

**Theorem 4.9.** *Let $\gamma \in T_I^{k^{\pm 1}}$. Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module. The map $\phi \mapsto \psi_\phi$, given by* (4.6)*, restricts to a linear isomorphism*

$$L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}] \xrightarrow{\sim} \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W.$$

*Proof.* Let $\gamma \in T_I^{k^{\pm 1}}$ and $\psi \in \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{W^a}$, written as

$$\psi = \sum_{w \in W_0^I} \psi_w \otimes v_w^{k,\pm,I}(\gamma)$$

with $\psi_w \in \pi^{k^{-1},q}\big(T_{w\overline{w_0}^{-1}}\big)\phi$ and $\phi \in L_{\pi,a}^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$, cf. Proposition 4.7. Then $\psi \in \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W$ if and only if

$$\sum_{w \in W_0^I} \pi^{k^{-1},q}(T_\omega)\psi_w \otimes v_w^{k,\pm,I}(\gamma) = \sum_{w \in W_0^I} \psi_w \otimes T_\omega^{-1} v_w^{k,\pm,I}(\gamma) \tag{4.8}$$

for all $\omega \in \Omega$ in view of Proposition 3.10.

In order to analyze (4.8), we first need to give some additional properties of the abelian subgroup $\Omega \subset W$ (we refer to [23] for detailed proofs of the following facts). For $\lambda \in P^\vee$ we write $v(\lambda)$ for the unique element of minimal length in $W_0$ such that $v(\lambda)\lambda \in w_0 P_+^\vee$. We furthermore set $u(\lambda) \in W$ such that $t(\lambda) = u(\lambda)v(\lambda)$ in $W$. Set $\varpi_0^\vee \in 0$. Then we write, for $0 \leq j \leq n$, $u_j = u(\varpi_j^\vee) \in W$ and $v_j = v(\varpi_j^\vee)$. In particular, $u_0 = e$ and $v_0 = e$. Set

$$J := \{0\} \cup \{j \in \{1, \ldots, n\} \mid \langle \varpi_j^\vee, \varphi\rangle = 1\}.$$

Then $\Omega = \{u_j\}_{j \in J}$. We write $\{U_j\}_{j \in J}$ for the corresponding elements in the extended affine Hecke algebra $H(k)$. Then in $H(k)$,

$$U_j = T_w Y^{w^{-1}(\varpi_j^\vee)} T_{v_j w}^{-1} \qquad (j \in J,\, w \in W_0), \tag{4.9}$$

cf. [23, (3.3.3)].

Choose $\sigma_j \in W_0$ $(j \in J)$ arbitrarily. Then $\{\sigma_j(\varpi_j^\vee)\}_{j \in J}$ is a complete set of representatives of $P^\vee/Q^\vee$. In particular, the standard parabolic subalgebra $H_I(k)$ of $H(k)$ is generated by $H_I^a(k)$ and the $Y^{\sigma_j(\varpi_j^\vee)}$ $(j \in J)$.

We are now set to analyze (4.8). Let $j\in J$ and $w\in W_0^I$, then it follows from (4.9) that

$$U_j^{-1}v_w^{k,\pm,I}(\gamma)=\epsilon_\pm^k(T_{w_j})\gamma^{-w^{-1}(\varpi_j^\vee)}v^{k,\pm,I}_{\overline{v_jw}}(\gamma),$$

where we have written $v_jw=(\overline{v_jw})w_j$ with $\overline{v_jw}\in W_0^I$ and $w_j:=\underline{v_jw}\in W_{0,I}$. Note that $w\mapsto\overline{v_jw}$ defines a bijection $W_0^I\to W_0^I$ with inverse given by $w\mapsto\overline{v_j^{-1}w}$ ($w\in W_0^I$). We write $w_j':=\underline{v_j^{-1}w}\in W_{0,I}$, such that $v_j^{-1}w=(\overline{v_j^{-1}w})w_j'$. Note that

$$(\overline{v_j^{-1}w})_j=w_j'^{-1},\qquad w\in W_0^I \tag{4.10}$$

since $v_j(\overline{v_j^{-1}w})=ww_j'^{-1}$. Thus we conclude that

$$\sum_{w\in W_0^I}\psi_w\otimes U_j^{-1}v_w^{k,\pm,I}(\gamma)=\sum_{w\in W_0^I}\epsilon_\pm^k\big(T_{w_j'}\big)\gamma^{-(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)}\psi_{\overline{v_j^{-1}w}}\otimes v_w^{k,\pm,I}(\gamma).$$

Hence (4.8) holds true for all $\omega\in\Omega$ if and only if, for all $w\in W_0^I$ and $j\in J$,

$$\pi^{k^{-1},q}(U_j)\psi_w=\epsilon_\pm^k(T_{w_j'})\gamma^{-(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)}\psi_{\overline{v_j^{-1}w}}. \tag{4.11}$$

As in the proof of Proposition 4.7, it follows from (4.9) and $\psi_w=\pi^{k^{-1},q}(T_{w^{-1}}^{-1}T_{\overline{w_0}^{-1}})\phi$ that for all $w\in W_0^I$,

$$\pi^{k^{-1},q}(U_j)\psi_w=\pi^{k^{-1},q}\big(T^{-1}_{(\overline{v_j^{-1}w})^{-1}}T^{-1}_{w_j'^{-1}}T_{\underline{w_0}}T_{\overline{w_0}^{-1}}Y^{w_0w^{-1}v_j(\varpi_j^\vee)}T^{-1}_{\overline{w_0}^{-1}}T^{-1}_{\underline{w_0}}T_{\overline{w_0}^{-1}}\big)\phi.$$

Using that $\phi\in L_{\pi,a}^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$, in particular $\pi^{k^{-1},q}\big(T_{\overline{w_0}^{-1}}\big)\phi\in L_\pi^{I,\pm}$, we get $\psi\in\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)^W_\nabla$ iff (4.11) holds true for all $w\in W_0^I$ and $j\in J$, iff

$$\pi^{k^{-1},q}\big(Y^{w_0w^{-1}v_j(\varpi_j^\vee)}\big)\phi=\gamma^{-(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)}\phi\quad\forall\,w\in W_0^I,\ \forall\,j\in J.$$

It thus remains to prove that for $\gamma\in T_I^k$,

$$\gamma^{-(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)}=\gamma_I^{w_0w^{-1}v_j(\varpi_j^\vee)}\quad\forall\,w\in W_0^I,\ \forall\,j\in J, \tag{4.12}$$

with $\gamma_I$ given by (4.7).

Now (4.12) for $\gamma\in T_I^k$ is equivalent to

$$\gamma^{(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)}=(\rho_I^k)^{w^{-1}v_j(\varpi_j^\vee)}\gamma^{w^{-1}v_j(\varpi_j^\vee)}\quad\forall\,w\in W_0^I,\ \forall\,j\in J.$$

Since $\gamma\in T_I^k$ and $w^{-1}v_j=w_j'^{-1}(\overline{v_j^{-1}w})^{-1}$ for $w\in W_0^I$ we have by (2.16),

$$\gamma^{w^{-1}v_j(\varpi_j^\vee)}=\gamma^{(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)}\prod_{\alpha\in R_0^{I,+}\cap w_j'R_0^{I,-}}k_{\alpha^\vee}^{-2\langle\alpha,(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)\rangle},$$

while, by the definition of $\rho_I^k$,

$$(\rho_I^k)^{w^{-1}v_j(\varpi_j^\vee)}=\prod_{\alpha\in R_0^{I,+}}k_{\alpha^\vee}^{-2\langle\alpha,w^{-1}v_j(\varpi_j^\vee)\rangle}=\prod_{\alpha\in w_j'(R_0^{I,-})}k_{\alpha^\vee}^{2\langle\alpha,(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)\rangle}.$$

Hence (4.12) holds true iff for all $w\in W_0^I$ and $j\in J$,

$$\langle\alpha,(\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)\rangle=0,\quad\forall\,\alpha\in R_0^{I,-}\cap w_j'(R_0^{I,-}). \tag{4.13}$$

Fix $w \in W_0^I$ and $j \in J$. We show in fact that

$$\langle \alpha, (\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)\rangle \begin{cases} \leq 0, & \forall\, \alpha \in R_0^{I,-}, \\ \geq 0, & \forall\, \alpha \in w_j'(R_0^{I,-}), \end{cases} \tag{4.14}$$

which implies (4.13). The first inequality is immediate, since $u(R_0^{I,-}) \subseteq R_0^-$ if $u \in W_0^I$. For the second equality, let $\alpha = w_j'(\beta) \in w_j'(R_0^{I,-})$. Then, since $(\overline{v_j^{-1}w})w_j' = v_j^{-1}w$,

$$\langle \alpha, (\overline{v_j^{-1}w})^{-1}(\varpi_j^\vee)\rangle = \langle w(\beta), v_j(\varpi_j^\vee)\rangle.$$

But $w \in W_0^I$ and $\beta \in R_0^{I,-}$ hence $w(\beta) \in R_0^-$, and $v_j(\varpi_j^\vee) \in w_0P_+^\vee$, hence $\langle w(\beta), v_j(\varpi_j^\vee)\rangle \geq 0$. This concludes the proof. □

**Corollary 4.10.** *Let $\gamma \in T_I^{k^{\pm 1}}$ and let $L$ a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module. The we have an injective linear map*

$$\alpha_{L,\gamma}^{k,q,\pm,I} : \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W \hookrightarrow \Gamma_L^{k,q}\big(M^k(\underline{w_0}\gamma)\big)_\nabla^W,$$

*defined by*

$$\alpha_{L,\gamma}^{k,q,\pm,I}\big(\sum_{w\in W_0^I}\psi_w \otimes v_w^{k,\pm,I}(\gamma)\big) = \sum_{u\in W_0^I}\psi_u \otimes \Big(\sum_{v\in W_{I,0}}\epsilon_\pm^k(T_v)v_{uv}^k(\underline{w_0}\gamma)\Big).$$

*Proof.* We claim that $\epsilon_\pm^k(T_{\underline{w_0}})^{-1}\alpha_{L,\gamma}^{k,q,\pm,I}$ is equal to the composition of maps

$$\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W \xrightarrow{\sim} L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}] \hookrightarrow L_{\pi,\overline{w_0}\gamma^{-1}} \xrightarrow{\sim} \Gamma_L^{k,q}\big(M^k(\underline{w_0}\gamma)\big)_\nabla^W,$$

with the second the trivial inclusion, and the first and third isomorphisms obtained from Theorem 4.9. This can be proved by a direct computation, using the fact that

$$\pi^{k^{-1},q}(T_{w^{-1}}^{-1}T_{w_0})\phi = \epsilon_\pm^k(T_{\underline{w_0}})^{-1}\epsilon_\pm^k(T_{\underline{w}})\pi^{k^{-1},q}(T_{\overline{w}^{-1}}^{-1}T_{\overline{w_0}^{-1}})\phi$$

for $\phi \in L_\pi^{I^*,\pm}$ and $w \in W_0$. □

Let $\gamma \in T_I^{k^{\pm 1}}$ and $L$ a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module. The functoriality of the assignment (3.13) can be applied to the surjective morphism $M^k(\gamma) \to M^{k,\pm,I}(\gamma)$ of $H(k)$-modules mapping $v_e^k(\gamma)$ to $v_e^{k,\pm,I}(\gamma)$. In fact, it maps $v_w^k(\gamma)$ to $\epsilon_\pm^k(T_{\underline{w}})v_{\overline{w}}^{k,\pm,I}(\gamma)$ for $w \in W_0$. It thus gives rise to a $\mathbb{C}[W]$-linear map

$$\begin{aligned} \Gamma_L^{k,q}(M^k(\gamma))_\nabla &\to \Gamma_L^{k,q}(M^{k,\pm,I}(\gamma))_\nabla, \\ \sum_{w\in W_0}\psi_w \otimes v_w^k(\gamma) &\mapsto \sum_{u\in W_0^I}\Big(\sum_{v\in W_{0,I}}\epsilon_\pm^k(T_v)\psi_{uv}\Big)\otimes v_u^{k,\pm,I}(\gamma) \end{aligned} \tag{4.15}$$

for $\gamma \in T_I^{k^{\pm 1}}$. Combined with Theorem 4.9 we obtain

**Corollary 4.11.** *Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$ and let $\gamma \in T_I^{k^{\pm 1}}$. The assignment*

$$\phi \mapsto \sum_{v\in W_{0,I^*}}\epsilon_\pm^{k^{-1}}(T_v)\pi^{k^{-1},q}(T_v)\phi$$

*defines a linear map*

$$L_{\pi,w_0\gamma^{-1}} \to L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}].$$

*Proof.* Let $\gamma\in T_I^{k^{\pm 1}}$. Denote by
$$\eta_\gamma^I: L_{\pi,w_0\gamma^{-1}}\rightarrow L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$$
the composition of the linear maps
$$L_{\pi,w_0\gamma^{-1}}\overset{\sim}{\longrightarrow}\Gamma_L^{k,q}(M^k(\gamma))_\nabla^W\rightarrow\Gamma_L^{k,q}(M^{k,\pm,I}(\gamma))_\nabla^W\overset{\sim}{\longrightarrow}L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}],$$
where the isomorphisms are from Theorem 4.9, and the second map is the restriction of (4.15) to $\Gamma_L^{k,q}(M^k(\gamma))_\nabla^W$. Then
$$\eta_\gamma^I(\phi)=\sum_{v\in W_{0,I}}\epsilon_\pm^k(T_v)\pi^{k^{-1},q}(T_{\overline{w_0}^{-1}}^{-1}T_{v^{-1}}^{-1}T_{w_0})\phi$$
for $\phi\in L_{\pi,w_0\gamma^{-1}}$. It suffices to show that
$$\eta_\gamma^I(\phi)=\epsilon_\pm^k(T_{\underline{w_0}})\sum_{v\in W_{0,I^*}}\epsilon_\pm^{k^{-1}}(T_v)\pi^{k^{-1},q}(T_v)\phi\qquad(\phi\in L_{\pi,w_0\gamma^{-1}}).\tag{4.16}$$
Let $\phi\in L_{\pi,w_0\gamma^{-1}}$. Note that $W_{0,I}=\overline{w_0}^{-1}W_{0,I^*}\overline{w_0}$, and
$$T_{\overline{w_0}^{-1}v\overline{w_0}}=T_{\overline{w_0}^{-1}}T_vT_{\overline{w_0}^{-1}}^{-1}\qquad(v\in W_{0,I^*}).$$
It follows that
$$\eta_\gamma^I(\phi)=\sum_{v\in W_{0,I^*}}\epsilon_\pm^k(T_v)\pi^{k^{-1},q}(T_{v^{-1}}^{-1}T_{\overline{w_0}^{-1}}^{-1}T_{w_0})\phi.$$
Denote $\underline{w_0}^*$ for the longest Weyl group element of $W_{0,I^*}$. Then $\underline{w_0}^*=\overline{w_0}\underline{w_0}\overline{w_0}^{-1}$, hence
$$T_{\overline{w_0}^{-1}}^{-1}T_{w_0}=T_{\overline{w_0}w_0}=T_{\underline{w_0}^*}.$$
It follows that
$$\eta_\gamma^I(\phi)=\sum_{v\in W_{0,I^*}}\epsilon_\pm^k(T_v)\pi^{k^{-1},q}(T_{v\underline{w_0}^*})\phi.$$
Using that $T_{v\underline{w_0}^*}=T_{v^{-1}}^{-1}T_{\underline{w_0}^*}$ for $v\in W_{0,I^*}$, we get (4.16). □

4.3. **GL$_m$ case.** It is instructive to consider the GL$_m$ case of Theorem 4.9, since it has a simpler proof. We keep the notations and definitions as before with $R_0$ the root system of type $A_{m-1}$ ($m\geq 2$). We redefine first, for the duration of this subsection, those notions from the previous subsections which need slight modifications in the GL$_m$ setup.

Let $\{\epsilon_i\}_{i=1}^m$ be an orthonormal basis of $\mathbb{R}^m$. Take $\{\epsilon_i-\epsilon_j\}_{1\leq i<j\leq m}$ as the realization of the root system $R_0$, and take $\alpha_j=\epsilon_j-\epsilon_{j+1}$ ($1\leq j\leq n=m-1$) as a basis of $R_0$. The affine root system is $R=\mathbb{Z}c+R_0$ with additional affine simple root $a_0=c-\epsilon_1+\epsilon_m$.

Set $T=\big(\mathbb{C}\setminus\{0\}\big)^m$ and define for $\lambda\in\mathbb{Z}^m$ and $r\in\mathbb{Z}$, the monomial $e_q^{rc+\lambda}\in\mathbb{C}[T]$ by
$$e_q^{rc+\lambda}(t)=q^rt^\lambda,\qquad t\in T$$
(where we use the usual multi-index notations). The extended affine Weyl group is $W=S_m\ltimes\mathbb{Z}^m$. A multiplicity function $R\rightarrow\mathbb{C}^\times$ (i.e. $W$-invariant) takes on a constant value, which we denote by $k\in\mathbb{C}^\times$. We let $W$ act on $\mathbb{C}[T]$ by $q$-difference reflection operators (see (3.1)).

The affine Weyl group $W$ is generated by $s_i=s_{\alpha_i}$ ($1\leq i<m$) and $\zeta$, where
$$\zeta=\sigma\tau(\epsilon_m)$$

with $\sigma = s_1 s_2 \cdots s_{m-1} \in S_m$. It thus acts on $f \in \mathbb{C}[T]$ by

$$(\zeta_q f)(t) = f(t_2, \ldots, t_m, q^{-1}t_1).$$

Let $\Omega \subset W$ be the subgroup of $W$ generated by $\zeta$. It is the subgroup consisting of the affine Weyl group elements $w \in W$ of length zero.

The extended affine Hecke algebra $H(k)$ is generated by $T_\zeta$ and $T_1, \ldots, T_{m-1}$ with, besides the familiar relations for the finite Hecke algebra generators $\{T_i\}_{1\leq i<m}$ (including the quadratic relations $(T_i - k)(T_i + k^{-1}) = 0$), the relations that $T_\zeta^m$ is central in $H(k)$ and that $T_\zeta T_i = T_{i+1}T_\zeta$ for $1 \leq i < m-1$. The commuting $Y_i = Y^{\epsilon_i} \in H(k)$ $(1 \leq i \leq m)$ are given by

$$Y_i = T_{i-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_\zeta T_{m-1} T_{m-2} \cdots T_i.$$

Let $\mathcal{A}_Y^k$ be the commutative subalgebra of $H(k)$ generated by the $Y_i^{\pm 1}$ $(1 \leq i \leq m)$. The extended affine Hecke algebra is generated by the finite Hecke algebra $H_0(k)$ (which is generated by $T_1, \ldots, T_{m-1}$) and $\mathcal{A}_Y^k$, with defining relations as in Theorem 2.3.

The algebra map $\pi^{k,q} : H(k) \to \mathbb{C}_\sigma^q[T]\#_q W$ is now given by

$$\pi^{k,q}(T_i) = k + c_i^{k,q}(s_i - 1), \qquad \pi^{k,q}(T_\zeta) = \zeta$$

for $1 \leq i < m$. The algebra map $\nabla^{k,q} : \mathbb{C}[T]\#_q W \to \mathcal{A}_\nabla^{k,q}$ is

$$\begin{aligned} \nabla^{k,q}(f) &= f, \\ \nabla^{k,q}(s_i) &= (c_i^{k,q})^{-1} s_i T_i + \frac{c_i^{k,q} - k}{c_i^{k,q}} s_i, \\ \nabla^{k,q}(\zeta) &= \zeta T_\zeta \end{aligned}$$

for $f \in \mathbb{C}[T]$ and $1 \leq i < m$.

For $I \subseteq \{1, \ldots, m-1\}$ we write $H_I(k)$ for the subalgebra generated by $T_i$ $(i \in I)$ and $\mathcal{A}_Y^k$. For $\gamma \in T_I^{k^{\pm 1}}$ we have a character $\chi_\gamma^{k,\pm,I} : H_I(k) \to \mathbb{C}$ mapping $T_i$ to $\pm k^{\pm 1}$ for $i \in I$ and $f(Y)$ to $f(\gamma)$ for $f \in \mathbb{C}[T]$. We have the associated affine Hecke algebra module $M^{k,\pm,I}(\gamma) = \mathrm{Ind}_{H_I(k)}^{H(k)}\big(\chi_\gamma^{k,\pm,I}\big)$ with basis $v_w^{k,\pm,I}(\gamma)$ $(w \in S_m^I)$, where $S_m^I$ is the set of minimal coset representatives of $S_m/S_{m,I}$ (with $S_{m,I}$ the subgroup of $S_m$ generated by $s_i$ $(i \in I)$).

Let $w_0 \in S_m$ be the longest Weyl group element, mapping $j$ to $m+1-j$ for $1 \leq j \leq m$, and set

$$\varpi_j = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_j, \qquad 1 \leq j \leq m.$$

Corollary 4.4 now holds true in the present $\mathrm{GL}_m$ setup.

**Proposition 4.12.** *Let $\gamma \in T_I^{k^{\pm 1}}$ and let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module. Suppose that $\phi \in L_\pi^{I^*,\pm}$ and let $\psi_\phi \in \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{S_m}$ be defined by (4.6). Then $\nabla^{k,q}(\tau(\varpi_j))\psi_\phi$ is equal to*

$$\sum_{w \in S_m^I} \epsilon_\pm^k(T_{\underline{w_0}} T_{w_j'}) \gamma^{w_j' w^{-1}(\varpi_j)} \pi^{k^{-1},q}\big(T_{w w_j'^{-1} w_0} Y^{w_0 w_j' w^{-1}(\varpi_j)}\big)\phi \otimes v_w^{k,I}(\gamma)$$

*for $1 \leq j \leq m$, where $w_j' \in S_{m,I}$ is such that $\sigma^{-j} w w_j'^{-1} \in S_m^I$.*

*Proof.* Note that $\tau(\varpi_j)=\zeta^j\sigma^{-j}\in W$, hence
$$\nabla^{k,q}(t(\varpi_j))\psi_\phi=\nabla^{k,q}(\zeta^j)\psi_\phi.$$
For $\phi\in L_\pi^{I^*,\pm}$ we have $\pi^{k^{-1},q}(T_{\overline{w_0}^{-1}})\phi\in L_\pi^{I,\pm}$, hence for all $w\in S_m^I$,
$$\pi^{k^{-1},q}(T_{w\overline{w_0}^{-1}})\phi=\pi^{k^{-1},q}(T_{w^{-1}}^{-1}T_{\overline{w_0}^{-1}})\phi=\epsilon_\pm^k(T_{\underline{w_0}})\pi^{k^{-1},q}(T_{ww_0})\phi.$$
Thus $\psi_\phi=\sum_{w\in S_m^I}\psi_w\otimes v_w^{k,\pm,I}(\gamma)$ with $\psi_w=\epsilon_\pm^k(T_{\underline{w_0}})\pi^{k^{-1},q}(T_{ww_0})\phi$. Using the fact that
$$T_\zeta T_u=T_{\sigma u}Y_{u^{-1}(m)},\qquad u\in S_m \tag{4.17}$$
in $H(k)$ (cf., e.g., the proof of [27, Lemma 4.1]), we obtain
$$\begin{aligned}\nabla^{k,q}(\tau(\varpi_j))\psi_\phi&=\nabla^{k,q}(\zeta^j)\psi_\phi\\&=\sum_{w\in S_m^I}\zeta^j\psi_w\otimes T_\zeta^jv_w^{k,\pm,I}(\gamma)\\&=\sum_{w\in S_m^I}\epsilon_\pm^k(T_{\underline{w_0}}T_{w_j})\gamma^{w^{-1}w_0(\varpi_j)}\pi^{k^{-1},q}(T_\zeta^jT_{ww_0})\phi\otimes v^{k,\pm,I}_{\overline{\sigma^jw}}(\gamma),\end{aligned}$$
where $w_j:=\underline{\sigma^jw}\in S_{m,I}$, such that $\sigma^jw=(\overline{\sigma^jw})w_j$.

For $w\in S_m^I$ we write $w_j':=\underline{\sigma^{-j}w}\in S_{m,I}$, such that $\sigma^{-j}w=(\overline{\sigma^{-j}w})w_j'$. Then $w\mapsto\overline{\sigma^jw}$ defines a bijection $S_m^I\xrightarrow{\sim}S_m^I$, with inverse $w\mapsto\overline{\sigma^{-j}w}$ ($w\in S_m^I$). Furthermore, $(\overline{\sigma^{-j}w})_j=w_j'^{-1}$ for $w\in S_m^I$. Thus
$$\begin{aligned}&\nabla^{k,q}(\tau(\varpi_j))\psi_\phi=\\&=\sum_{w\in S_m^I}\epsilon_\pm^k(T_{\underline{w_0}}T_{w_j'})\gamma^{w_j'w^{-1}\sigma^jw_0(\varpi_j)}\pi^{k^{-1},q}(T_\zeta^jT_{\sigma^{-j}ww_j'^{-1}w_0})\phi\otimes v_w^{k,\pm,I}(\gamma)\\&=\sum_{w\in S_m^I}\epsilon_\pm^k(T_{\underline{w_0}}T_{w_j'})\gamma^{w_j'w^{-1}(\varpi_j)}\pi^{k^{-1},q}\big(T_{ww_j'^{-1}w_0}Y^{w_0w_j'w^{-1}(\varpi_j)}\big)\phi\otimes v_w^{k,\pm,I}(\gamma),\end{aligned}$$
where we have used (4.17) and $\sigma^jw_0(\varpi_j)=\varpi_j$ to obtain the last equality. □

*Remark* 4.13. The classical ($q=1$) analogue of Proposition 4.12 is [28, Lemma 3.2]. In contrast to the present $q$-setup, it is for arbitrary root systems.

We write $\rho_I^k=(\rho_{I,1}^k,\ldots,\rho_{I,m}^k)\in T$ with
$$\rho_{I,j}^k=\prod_{\alpha\in R_0^{I,+}}k^{-2\langle\epsilon_j,\alpha\rangle}.$$
For $\gamma\in T_I^k$ we have $\overline{w_0}\gamma^{-1}=(w_0\rho_I^k)^{-1}(w_0\gamma)^{-1}\in T_{I^*}^{k^{-1}}$ (cf. Lemma 4.5 and (4.7)). As before, we define for $\gamma\in T_I^{k^{\pm1}}$ and for a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module $L$,
$$L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]=\{\phi\in L_\pi^{I^*,\pm}\mid\pi^{k^{-1},q}(f(Y))\phi=f(\overline{w_0}\gamma^{-1})\phi\quad\forall f\in\mathbb{C}[T]\}.$$
The $\mathrm{GL}_m$ version of Theorem 4.9 which we now formulate, is essentially due to Kasatani and Takeyama [17] (the present version is more general since we do not need to impose any parameter restraints). We give a proof based on Proposition 4.12.

**Theorem 4.14.** *Let $\gamma\in T_I^{k^{\pm1}}$ and let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module. Then $\phi\mapsto\psi_\phi$ defines a linear bijection $L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]\xrightarrow{\sim}\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W$.*

*Proof.* Let $\gamma\in T_I^{k^{\pm 1}}$ and $\phi\in L_\pi^{I^*,\pm}$. Since for $w\in S_m^I$ and $1\leq j\leq m$,

$$T_{ww_j'^{-1}w_0}=T_{w^{-1}}^{-1}T_{w_j'}^{-1}T_{\underline{w_0}}T_{\overline{w_0}^{-1}},$$

it follows from Proposition 4.12 and $\pi^{k^{-1},q}(T_{\overline{w_0}^{-1}})\phi\in L_\pi^{I,\pm}$ that $\psi_\phi\in\Gamma_L\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{S_m}$ is $W$-invariant if and only of

$$\pi^{k^{-1},q}\big(Y^{w_0w_j'w^{-1}(\varpi_j)}\big)\phi=\epsilon_\pm^k(T_{w_j'})^{-2}\gamma^{-w_j'w^{-1}(\varpi_j)}\phi$$

for all $w\in S_m^I$ and $1\leq j\leq m$. Since $\overline{w_0}\gamma^{-1}=w_0(\rho_I^{k^{\pm 1}}\gamma)^{-1}$, it thus suffices to show that

$$(\rho_I^k)^{w_j'w^{-1}(\varpi_j)}=\epsilon_+^k(T_{w_j'})^2$$

for all $1\leq j\leq m$ and $w\in S_m^I$. Fix $w\in S_m^I$ and $1\leq j\leq m$. Then

$$(\rho_I^k)^{w_j'w^{-1}(\varpi_j)}=\prod_{\alpha\in w_j'^{-1}(R_0^{I,-})}k^{2\langle\alpha,w^{-1}(\varpi_j)\rangle},\quad \epsilon_+^k(T_{w_j'})^2=\prod_{\alpha\in R_0^{I,+}\cap w_j'^{-1}R_0^{I,-}}k^2,$$

so it remains to show that

$$\langle\alpha,w^{-1}(\varpi_j)\rangle=\begin{cases}1 & \text{if }\ \alpha\in R_0^{I,+}\cap w_j'^{-1}R_0^{I,-},\\ 0 & \text{if }\ \alpha\in R_0^{I,-}\cap w_j'^{-1}R_0^{I,-}.\end{cases}\tag{4.18}$$

First note that if $\alpha=w_j'^{-1}\beta$ ($\beta\in R_0^{I,-}$) then

$$\langle\alpha,w^{-1}(\varpi_j)\rangle=\langle(\overline{\sigma^{-j}w})(\beta),\sigma^{-j}(\varpi_j)\rangle\geq 0$$

since $\sigma^{-j}(\varpi_j)=w_0(\varpi_j)$ and $u(R_0^{I,-})\subseteq R_0^-$ for $u\in S_m^I$. But $w_0(\varpi_j)$ is miniscule (i.e. $|\langle\alpha,w_0(\varpi_j)\rangle|\leq 1$ for all $\alpha\in R_0$), hence $\langle\alpha,w^{-1}(\varpi_j)\rangle\in\{0,1\}$ if $\alpha\in w_0'^{-1}(R_0^{I,-})$.

Fix now $\alpha\in w_j'^{-1}(R_0^{I,-})$. Suppose first that $\alpha\in R_0^{I,-}$. Then $\langle\alpha,w^{-1}(\varpi_j)\rangle=\langle w(\alpha),\varpi_j\rangle\leq 0$, since $w(\alpha)\in R_0^-$. On the other hand, we have already observed that the scalar product is 0 or 1, hence this forces $\langle\alpha,w^{-1}(\varpi_j)\rangle=0$. If on the other hand $\alpha\in R_0^{I,+}$, then, since $\sigma^{-j}w=(\overline{\sigma^{-j}w})w_j'$ and $l(\sigma^{-j}w)=l(\overline{\sigma^{-j}w})+l(w_j')$,

$$R_0^+\cap(\sigma^{-j}w)^{-1}R_0^-=w_j'^{-1}\big(R_0^+\cap(\overline{\sigma^{-j}w})^{-1}R_0^-\big)\cup(R_0^{I,+}\cap w_j'^{-1}R_0^{I,-})$$

(disjoint union), thus $\alpha\in R_0^+\cap(\sigma^{-j}w)^{-1}R_0^-$. Since furthermore $\alpha\in R_0^{I,+}$ and $w\in S_m^I$ we have $w(\alpha)\in R_0^+$, hence $w(\alpha)\in R_0^+\cap\sigma^jR_0^-$. But

$$\begin{aligned}R_0^+\cap\sigma^jR_0^-&=R^+\cap\sigma^jR^-\\&=R^+\cap\sigma^j\zeta^{-j}R^-\\&=R^+\cap\tau(-\varpi_j)R^-.\end{aligned}$$

In other words,

$$\tau(\varpi_j)(w\alpha)=w\alpha-\langle\varpi_j,w\alpha\rangle c\in R^-.\tag{4.19}$$

We already observed that $w\alpha\in R_0^+$ and that $\langle\varpi_j,w\alpha\rangle$ is 0 or 1, hence (4.19) can only hold true if $\langle\varpi_j,w\alpha\rangle=1$. This completes the proof of (4.18), and hence of the theorem. □

## 5. Cherednik-Matsuo type correspondences

We return now again to arbitrary root systems. Cherednik [4] related solutions of quantum affine KZ equations with values in principal series modules to common eigenfunctions of the commuting Cherednik-Macdonald scalar $q$-difference operators. We discuss and deepen this result using the results of the previous sections.

5.1. **(Anti)symmetrization.** By Lemma 2.15 we have for $\gamma \in T_I^{k^{\pm 1}}$ a linear map

$$\pi^{k^{-1},q}(C_{\pm}^{I^*}(k^{-1})) : L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}] \to \big(L_\pi^{W_0\gamma^{-1}}\big)^{\pm}, \tag{5.1}$$

which we call the symmetrization $(+)$, respectively antisymmetrization $(-)$, map. Recall here that $L_\pi^{W_0\gamma^{-1}}$ is the $H(k^{-1})$-submodule

$$L_\pi^{W_0\gamma^{-1}} = \{\phi \in L \mid \pi^{k^{-1},q}(f(Y))\phi = f(\gamma^{-1})\phi \quad \forall f \in \mathbb{C}[T]^{W_0}\}$$

of $L_\pi$. Define

$$T_{I,reg}^k := \{\gamma \in T_I^k \mid k_{\alpha^\vee}^2 \neq \gamma^{\alpha^\vee} \neq 1 \quad \forall \alpha \in R_0^+ \setminus R_0^{I,+}\}. \tag{5.2}$$

**Proposition 5.1.** *Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module.*
**(a)** *If $\gamma \in T_{I,reg}^{k^{\pm 1}}$ then the (anti)symmetrization map* (5.1) *is injective,*

$$\pi^{k^{-1},q}(C_{\pm}^{I^*}(k^{-1})) : L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}] \hookrightarrow \big(L_\pi^{W_0\gamma^{-1}}\big)^{\pm}.$$

**(b)** *Let $\gamma \in T$ such that $k_{\alpha^\vee}^2 \neq \gamma^{\alpha^\vee} \neq 1$ for all $\alpha \in R_0$. Then*

$$\pi^{k^{-1},q}(C_{\pm}(k^{-1})) : L_{\pi,w_0\gamma^{-1}} \xrightarrow{\sim} \big(L_\pi^{W_0\gamma^{-1}}\big)^{\pm}$$

*is a linear isomorphism.*

*Proof.* **(a)** For $\gamma \in T_I^k$ we have $\gamma_I := \overline{w_0}\gamma^{-1} \in T_{I^*}^{k^{-1}}$, cf. Lemma 4.5. Furthermore,

$$\overline{w_0}(R_0^- \setminus R_0^{I,-}) = R_0^+ \setminus R_0^{I^*,+}.$$

Hence, for $\gamma \in T$, we have $\gamma \in T_{I,reg}^k$ if and only if $\gamma_I \in T_{I^*}^{k^{-1}}$ and $k_{\alpha^\vee}^2 \neq \gamma_I^{\alpha^\vee} \neq 1$ for all $\alpha \in R_0^+ \setminus R_0^{I^*,+}$.

Let $\gamma \in T_{I,reg}^{k^{\pm 1}}$ and $\phi \in L_\pi^{I^*,\pm}[\gamma_I]$, and suppose that $\pi^{k^{-1},q}(C_{\pm}^{I^*}(k^{-1}))\phi = 0$. Since $\phi \in L_{\pi,\gamma_I}$ we have $I_w(k^{-1})\phi \in L_{\pi,w\gamma_I}$ for $w \in W_0^{I^*}$. The condition $\gamma_I^{\alpha^\vee} \neq 1$ for all $\alpha \in R_0^+ \setminus R_0^{I^*,+}$ implies that the $w\gamma_I$ ($w \in W_0^{I^*}$) are pairwise different (see Proposition 2.12(**i**)). Hence $\pi^{k^{-1},q}(C_{\pm}^{I^*}(k^{-1}))\phi = 0$ implies, in view of Theorem 2.18, that

$$\Big(\prod_{\alpha \in R_0^+ \setminus R_0^{I^*,+}} (\pm k_{\alpha^\vee}^{\pm 1} \mp k_{\alpha^\vee}^{\mp 1}\gamma_I^{\alpha^\vee})\Big)\phi = 0.$$

Since $\gamma_I^{\alpha^\vee} \neq k_{\alpha^\vee}^{\pm 2}$ for all $\alpha \in R_0^+ \setminus R_0^{I^*,+}$, we conclude that $\phi = 0$.
**(b)** Fix $\gamma \in T$ satisfying $k_{\alpha^\vee}^2 \neq \gamma^{\alpha^\vee} \neq 1$ for all $\alpha \in R_0$. By **(a)**, $\pi^{k^{-1},q}(C_{\pm}(k^{-1}))$ defines an injective linear map

$$\pi^{k^{-1},q}(C_{\pm}(k^{-1})) : L_{\pi,w_0\gamma^{-1}} \hookrightarrow \big(L_\pi^{W_0\gamma^{-1}}\big)^{\pm}. \tag{5.3}$$

It remains to show that (5.3) is surjective. For the remainder of the proof we leave out the map $\pi^{k^{-1},q}$ from the notations.

Choose a nonzero element $v \in \big(L_\pi^{W_0\gamma^{-1}}\big)^{\pm}$ and set $M := H(k^{-1})v$ for the associated cyclic $H(k^{-1})$-submodule of $L_\pi^{W_0\gamma^{-1}}$. By a result of Steinberg [34, Thm. 2.2],

$H(k^{-1})\simeq\mathcal{K}\otimes_{\mathbb{C}}\big(\mathcal{A}_Y^{k^{-1}}\big)^{W_0}\otimes_{\mathbb{C}}H_0(k^{-1})$ as complex vector spaces by the multiplication map, with $\mathcal{K}\subseteq\mathcal{A}_Y^{k^{-1}}$ a complex subspace of dimension $|W_0|$. It follows that $\mathrm{Dim}_{\mathbb{C}}\big(M\big)\leq|W_0|$.

For all $w\in W_0$, the intertwiner $I_{ww_0}(k^{-1})\in H(k^{-1})$ defines a linear bijection

$$I_{ww_0}(k^{-1}):M_{w_0\gamma^{-1}}\overset{\sim}{\longrightarrow}M_{w\gamma^{-1}}$$

with inverse $c_w^{-1}I_{w_0w^{-1}}(k^{-1})$, where (cf. Corollary 2.9),

$$c_w=d_{ww_0}(w\gamma^{-1})d_{w_0w^{-1}}(w_0\gamma^{-1}),$$

which is nonzero since $\gamma^{\alpha^\vee}\neq k_{\alpha^\vee}^2$ for all $\alpha\in R_0$. Since $M$ has central character $W_0\gamma^{-1}$, there exists a $u\in W_0$ such that $M_{u\gamma^{-1}}\neq 0$. We conclude that $M_{w\gamma^{-1}}\neq 0$ for all $w\in W_0$. Furthermore, $w\gamma^{-1}=w'\gamma^{-1}$ for $w,w'\in W_0$ iff $w=w'$, since $\gamma^{\alpha^\vee}\neq 1$ for all $\alpha\in R_0$ (see Proposition 2.12**(i)**). Combined with $\mathrm{Dim}_{\mathbb{C}}(M)\leq|W_0|$ we thus conclude that

$$M=\bigoplus_{w\in W_0}M_{w\gamma^{-1}},\qquad \mathrm{Dim}_{\mathbb{C}}\big(M_{w\gamma^{-1}}\big)=1\ \ \forall\, w\in W_0. \tag{5.4}$$

It follows from (5.4) and the conditions on $\gamma$ that $M\simeq M^{k^{-1}}(\gamma^{-1})$ as $H(k^{-1})$-modules. Hence, $M^{\pm}=\mathbb{C}v$ (cf. Lemma 2.17) and $C_{\pm}(k^{-1})$ defines a linear isomorphism

$$C_{\pm}(k^{-1}):M_{w_0\gamma^{-1}}\overset{\sim}{\longrightarrow}M^{\pm} \tag{5.5}$$

between the one-dimensional complex vector spaces $M_{w_0\gamma^{-1}}$ and $M^{\pm}=\mathbb{C}v$ of $M\subseteq L_\pi^{W_0\gamma^{-1}}$. We conclude that there exists a $m\in M_{w_0\gamma^{-1}}\subseteq L_{\pi,w_0\gamma^{-1}}$ such that $C_{\pm}(k^{-1})m=v$. Hence the map (5.3) is surjective. □

## 5.2. **The Cherednik-Matsuo map.**

**Definition 5.2.** *Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$ and $\gamma\in T_I^{k^{\pm1}}$. The Cherednik-Matsuo map $\xi_{L,\gamma}^{k,q,\pm,I}:\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)\to L$ is defined by*

$$\xi_{L,\gamma}^{k,q,\pm,I}\big(\sum_{w\in W_0^I}\psi_w\otimes v_w^{k,\pm,I}(\gamma)\big)=\sum_{w\in W_0^I}\epsilon_{\pm}^k(T_w)\psi_w.$$

*If $I=\emptyset$ then we write $\xi_{L,\gamma}^{k,q,\pm}=\xi_{L,\gamma}^{k,q,\pm,\emptyset}$.*

The map $\xi_{L,\gamma}^{k,q,+}$ was considered by Cherednik [2, 4]. Its classical analog appears in the work of Matsuo [25]. Observe that the Cherednik-Matsuo maps are compatible with the functorial maps $\Gamma_L^{k,q}(M^{k,\pm,I}(\gamma))\to\Gamma_L^{k,q}(M^{k,\pm,J}(\gamma))$ for $I\subseteq J$ and $\gamma\in T_J^{k^{\pm1}}$ (cf. (4.15) in the special case that $I=\emptyset$).

Note that the maps $\xi_{L,\gamma}^{k,q,\pm,I}$ for $\gamma\in T_I^{k^{\pm1}}$ are $W_0$-equivariant in the following sense,

$$\xi_{L,\gamma}^{k,q,\pm,I}\big(\nabla^{k,q}(w)\psi\big)=w_{\pm}\big(\xi_{L,\gamma}^{k,q,\pm,I}(\psi)\big)\qquad(\psi\in\Gamma_L^{k,q}(M^{k,\pm,I}(\gamma)),w\in W_0), \tag{5.6}$$

with $w_{\pm}\in\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$ given by (4.2). This follows directly from the explicit expression (3.7) of $\nabla^{k,q}(s_j)$ $(1\leq j\leq n)$ and (2.20). Recall the set $T_{I,reg}^k$ (see (5.2)) of regular elements in $T_I^k$.

**Proposition 5.3.** *Let $L$ be a left $\mathbb{C}^{k;q}_{\sigma,\nabla}[T]\#_q W$-module.*
**(a)** *Let $\gamma \in T_I^{k^{\pm 1}}$. Then $\xi^{k,q,\pm,I}_{L,\gamma}$ restricts to a linear map*

$$\xi^{k,q,\pm,I}_{L,\gamma} : \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W \to \big(L_\pi^{W_0\gamma^{-1}}\big)^\pm.$$

*The map is injective if $\gamma \in T^{k^{\pm 1}}_{I,reg}$.*
**(b)** *Let $\gamma \in T$ such that $k^2_{\alpha^\vee} \neq \gamma^{\alpha^\vee} \neq 1$ for all $\alpha \in R_0$. Then $\xi^{k,q,\pm}_{L,\gamma}$ restricts to a linear isomorphism*

$$\xi^{k,q,\pm}_{L,\gamma} : \Gamma_L^{k,q}\big(M^{k}(\gamma)\big)_\nabla^W \xrightarrow{\sim} \big(L_\pi^{W_0\gamma^{-1}}\big)^\pm.$$

*Proof.* By Theorem 4.9 and (5.1) we have for $\gamma \in T_I^{k^{\pm 1}}$ a linear map

$$\tilde{\xi}^{I,\pm}_\gamma : \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W \to \big(L_\pi^{W_0\gamma^{-1}}\big)^\pm \tag{5.7}$$

given by

$$\tilde{\xi}^{I,\pm}_\gamma\big(\sum_{w\in W_0^I}\psi_w \otimes v_w^{k,\pm,I}(\gamma)\big) = \sum_{w\in W_0^{I^*}} \epsilon_\pm^{k^{-1}}(T_w)\pi^{k^{-1},q}(T_w)\psi_{\overline{w_0}}.$$

By Theorem 4.9 and Proposition 5.1**(a)**, the map $\tilde{\xi}^{I,\pm}_\gamma$ is injective if $\gamma \in T^{k^{\pm 1}}_{I,reg}$. For $I=\emptyset$, by Theorem 4.9 and Proposition 5.1**(b)**, $\tilde{\xi}^{\emptyset,\pm}_\gamma$ is a linear isomorphism if $k^2_{\alpha^\vee} \neq \gamma^{\alpha^\vee} \neq 1$ for all $\alpha \in R_0$. To complete the proof of the theorem, it thus suffices to show that

$$\tilde{\xi}^{I,\pm}_\gamma(\psi) = \epsilon_\pm^{k^{-1}}(T_{\overline{w_0}})\xi^{k,q,\pm,I}_{L,\gamma}(\psi), \qquad \forall\, \psi \in \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W.$$

Let $\psi = \sum_{w\in W_0^I}\psi_w \otimes v_w^{k,\pm,I}(\gamma) \in \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W$. Then $\psi_w = \pi^{k^{-1},q}(T_{w\overline{w_0}^{-1}})\phi$ for all $w \in W_0^I$ with $\phi \in L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$.

Since $\overline{w_0}(\alpha_i) = \alpha_{i_I^*}$ for $i \in I$, we have $W_{0,I^*} = \overline{w_0}W_{0,I}\overline{w_0}^{-1}$ and $W_0^{I^*} = W_0^I\overline{w_0}^{-1}$. Furthermore, for $w \in W_0^I$ we have $T_{w\overline{w_0}^{-1}} = T^{-1}_{w^{-1}}T_{\overline{w_0}^{-1}}$, see the proof of Corollary 4.4. Hence we compute,

$$\begin{aligned}
\tilde{\xi}^{I,\pm}_\gamma(\psi) &= \sum_{w\in W_0^{I^*}} \epsilon_\pm^{k^{-1}}(T_w)\pi^{k^{-1},q}(T_w)\phi\\
&= \sum_{w\in W_0^I} \epsilon_\pm^{k^{-1}}(T_{w\overline{w_0}^{-1}})\pi^{k^{-1},q}(T_{w\overline{w_0}^{-1}})\phi\\
&= \sum_{w\in W_0^I} \epsilon_\pm^{k^{-1}}(T^{-1}_{w^{-1}}T_{\overline{w_0}^{-1}})\psi_w\\
&= \epsilon_\pm^{k^{-1}}(T_{\overline{w_0}})\sum_{w\in W_0^I} \epsilon_\pm^{k}(T_w)\psi_w\\
&= \epsilon_\pm^{k^{-1}}(T_{\overline{w_0}})\xi^{k,\gamma,\pm,I}_{L,\gamma}(\psi),
\end{aligned}$$

as desired. □

5.3. **Spectral problem of families of commuting $q$-difference operators.** For a left $\mathbb{C}^{k;q}_{\sigma,\nabla}[T]\#_q W$-module $L$ and $\gamma \in T$, we will identify, following Cherednik [4], the subspace $\big(L_\pi^{W_0\gamma^{-1}}\big)^\pm$, which appears in Proposition 5.3, with a space of common eigenfunctions of a commuting family of $q$-difference operators. In case of the symmetric theory (+), they are the Cherednik-Macdonald $q$-difference operators.

**Definition 5.4.** *The subalgebra*

$$\mathbb{D}_{\sigma,\nabla}^{k,q} := \mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q\tau(P^\vee)$$

*of* $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$ *is called the algebra of* $q$*-difference operators on* $T$ *with coefficients from* $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]$.

Recall the elements $w_\pm \in \mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$ from (4.2). A (twisted) $W_0$-action by algebra automorphisms on $\mathbb{D}_{\sigma,\nabla}^{k,q} \subset \mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$ is given by

$$(w, D) \mapsto w_\pm D w_\pm^{-1} \qquad (w \in W_0,\ D \in \mathbb{D}_{\sigma,\nabla}^{k,q}).$$

We set

$$\mathbb{D}_{\sigma,\nabla}^{k,q,\pm} := \{D \in \mathbb{D}_{\sigma,\nabla}^{k,q} \mid w_\pm D w_\pm^{-1} = D \quad \forall\, w \in W_0\}$$

the subalgebra of $\mathbb{D}_{\sigma,\nabla}^{k,q}$ consisting of $W_{0,\pm}$-invariant $q$-difference operators.

The following result is due to Cherednik [4].

**Proposition 5.5.** *For* $f \in \mathbb{C}[T]^{W_0}$ *write* $D_f^{k,q,\pm} := \sum_{w\in W_0} D_{f,w}^\pm$ *with* $D_{f,w}^\pm$ *the unique elements from* $\mathbb{D}_{\sigma,\nabla}^{k,q}$ *such that*

$$\pi^{k^{-1},q}(f(Y)) = \sum_{w\in W_0} D_{f,w}^\pm w_\pm$$

*in* $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$. *Then the map* $f \mapsto D_f^{k,q,\pm}$ *defines an algebra map*

$$\mathbb{C}[T]^{W_0} \to \mathbb{D}_{\sigma,\nabla}^{k,q,\pm}. \tag{5.8}$$

*In particular, the* $q$*-difference operators* $D_f^{k,q,\pm}$ *(*$f \in \mathbb{C}[T]^{W_0}$*) pairwise commute.*

For the proof of Proposition 5.5 one first shows that $D_f^{k,q,\pm}$ ($f \in \mathbb{C}[T]^{W_0}$) is $W_{0,\pm}$-invariant (for which one uses the fact that $f(Y)$ lies in the center of $H(k^{-1})$ if $f \in \mathbb{C}[T]^{W_0}$, as well as the expressions of $\pi^{k^{-1},q}(T_i)$ ($1 \leq i \leq n$) from the proof of Lemma 4.1). It then follows that (5.8) is an algebra homomorphism (see [21, Lemma 2.7] for a detailed proof in the symmetric case (+)).

**Definition 5.6.** *The* $D_f^{k,q,+} \in \mathbb{D}_{\sigma,\nabla}^{k,q,+}$ *(*$f \in \mathbb{C}[T]^{W_0}$*) are the Cherednik-Macdonald* $q$*-difference operators.*

The Macdonald $q$-difference operators correspond to $D_f^{k,q,+}$ with $f \in \mathbb{C}[T]^{W_0}$ given by $f(t) = \sum_{\mu\in W_0\lambda} t^\mu$ and $w_0\lambda$ a (quasi-)miniscule coweight. They can be written down explicitly (cf., e.g., [23, §4.4]).

**Definition 5.7.** *Let* $L$ *be a left* $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$*-module and* $\gamma \in T$. *We call*

$$\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma) := \{\phi \in L \mid D_f^{k,q,\pm}\phi = f(\gamma)\phi \quad \forall\, f \in \mathbb{C}[T]^{W_0}\}$$

*the solution space of the spectral problem of the commuting operators* $D_f^{k,q,\pm}$ *(*$f \in \mathbb{C}[T]^{W_0}$*) on* $L$*, with spectral parameter* $W_0\gamma \in T/W_0$.

By the previous proposition, $\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma)$ is a $W_{0,\pm}$-invariant subspace of $L$. We write $\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma)^{W_{0,\pm}}$ for the corresponding subspace of $W_{0,\pm}$-invariant elements.

**Proposition 5.8.** *Let $L$ be a $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW$-module and $\gamma\in T$. Then*

$$\big(L_\pi^{W_0\gamma}\big)^{\pm}=\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma)^{W_0,\pm}. \tag{5.9}$$

*Proof.* We have $L_\pi^{\pm}=L^{W_0,\pm}$ by Lemma 4.1 (where, recall, $L_\pi^{\pm}$ is the space of (anti)spherical vectors in $L_\pi$, see Subsection 2.6). Furthermore, for $\phi\in L^{W_0,\pm}$ we have, for all $f\in\mathbb{C}[T]^{W_0}$,

$$D_f^{k,q,\pm}\phi=\sum_{w\in W_0}D_{f,w}^{\pm}\phi=\sum_{w\in W_0}D_{f,w}^{\pm}w_{\pm}\phi=\pi^{k^{-1},q}(f(Y))\phi.$$

This implies now immediately (5.9). □

*Remark* 5.9. In this remark we explain the link to the theory of (anti)symmetric Macdonald polynomials (see [7, 23]). Fix $q\in\mathbb{C}^\times$, not a root of unity. Fix furthermore a multiplicity function $k$, sufficiently generic so that the theory of nonsymmetric Macdonald polynomials with respect to the parameters $(k^{-1},q)$ is to our proposal.

Set $L:=\mathbb{C}^{k,q}_{\sigma,\nabla}[T]$ (viewed as left $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW$-module) and consider the linear map

$$\pi^{k^{-1},q}(C_{\pm}(k^{-1})):L_\pi\to L_\pi^{\pm}$$

(see Lemma 2.15). It restricts, for each $\gamma\in T$, to a linear map

$$\pi^{k^{-1},q}(C_{\pm}(k^{-1})):L_\pi^{W_0\gamma^{-1}}\to\big(L_\pi^{W_0\gamma^{-1}}\big)^{\pm}.$$

Let $\lambda\in P^\vee$. Then there exists, up to a scalar multiple, a unique $0\neq E_\lambda(k^{-1},q)\in\mathbb{C}[T]\subset L$ satisfying

$$\pi^{k^{-1},q}(f(Y))E_\lambda(k^{-1},q)=f(\gamma_\lambda(k^{-1},q)^{-1})E_\lambda(k^{-1},q),\qquad\forall f\in\mathbb{C}[T],$$

where

$$\gamma_\lambda(k^{-1},q)=q^\lambda\prod_{\alpha\in R_0^+}k_{\alpha^\vee}^{-\eta(\langle\lambda,\alpha\rangle)\alpha}$$

with $\eta(x)=1$ if $x>0$ and $=-1$ if $x\leq 0$. The Laurent polynomial $E_\lambda(k^{-1},q)$ is the nonsymmetric Macdonald polynomial of degree $\lambda$. Let $P_+^\vee$ be the cone of dominant coweights. Then, writing $\gamma_\lambda=\gamma_\lambda(k^{-1},q)$, the symmetric and antisymmetric Macdonald polynomials are defined, for $\lambda\in P_+^\vee$, by

$$P_\lambda^{(\pm)}(k^{-1},q):=\pi^{k^{-1},q}(C_{\pm}(k^{-1}))E_\lambda(k^{-1},q),$$

which is an element in $\big(L_\pi^{W_0\gamma_\lambda^{-1}}\big)^{\pm}=\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma_\lambda^{-1})^{W_0,\pm}$. Under generic conditions on the parameters we have $P_\lambda^{(+)}(k^{-1},q)\neq 0$ for all $\lambda\in P_+^\vee$, and $P_\lambda^{(-)}(k^{-1},q)\neq 0$ unless $\lambda\in P_+^\vee\setminus(\rho^\vee+P_+^\vee)$, in which case $P_\lambda^{(-)}(k^{-1},q)=0$ (cf., e.g., [23, §5.7]).

From Proposition 5.3 we now immediately obtain the following important intermediate result.

**Proposition 5.10.** *Let $L$ be a left $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW$-module.*

**(a)** *Let $\gamma\in T_I^{k^{\pm1}}$. Then $\xi_{L,\gamma}^{k,q,\pm,I}$ restricts to a linear map*

$$\xi_{L,\gamma}^{k,q,\pm,I}:\Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W\to\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma^{-1})^{W_0,\pm}.$$

*The map is injective if* $\gamma \in T_{I,reg}^{k^{\pm 1}}$ *(see* (5.2)*).*

**(b)** *Let* $\gamma \in T$ *such that* $k_{\alpha^\vee}^2 \neq \gamma^{\alpha^\vee} \neq 1$ *for all* $\alpha \in R_0$*. Then* $\xi_{L,\gamma}^{k,q,\pm}$ *restricts to a linear isomorphism*

$$\xi_{L,\gamma}^{k,q,\pm} : \Gamma_L^{k,q}\big(M^k(\gamma)\big)_\nabla^W \xrightarrow{\sim} \mathrm{SpM}_L^{k,q,\pm}(W_0\gamma^{-1})^{W_0,\pm}.$$

*Remark* 5.11. Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module. Multiplcation by $G^{k,-}$ defines a linear isomorphism $L_\pi^+ \xrightarrow{\sim} L_\pi^-$. Through this map, in case $L = \mathbb{C}_{\sigma,\nabla}^{k,q}[T]$ and the parameters $k, q$ are generic, symmetric Macdonald polynomials are mapped to antisymmetric Macdonald polynomials with respect to a shifted multiplicity function, see, e.g., [23, (5.8.9)].

5.4. **The correspondences.** In this subsection we prove a nonsymmetric version of Proposition 5.10 (Proposition 5.10 then is reobtained by restriction to the subspace of $W_0$-invariant elements). We will prove the nonsymmetric version of the theorem using a construction which is motivated by Opdam's [28, §3] analysis of the trigonometric KZ equation (see also Cherednik and Ma [9, §3.5] for a different but closely related treatment).

Let $A$ and $B$ be unital associative $\mathbb{C}$-algebras. We write $\mathrm{BiMod}_{(A,B)}$ for the category of left $(A, B)$-bimodules over $\mathbb{C}$.

Let $A$ be an an unital associative $\mathbb{C}$-algebra, endowed with a left action of a group $G$ by unital algebra automorphisms. Write $A\#G$ for the associated smashed product algebra (so $A\#G \simeq A \otimes_{\mathbb{C}} \mathbb{C}[G]$ as vector spaces, with multiplication law $(a \otimes g)(a' \otimes g') = ag(a') \otimes gg'$). We then have a covariant functor

$$F_A^G : \mathrm{Mod}_{A\#G} \to \mathrm{BiMod}_{(A\#G,\mathbb{C}[G])} \tag{5.10}$$

defined as follows. Let $M$ be a left $A\#G$-module. Then $F_A^G(M)$ as complex vector space is the space of functions $f : G \to M$. It is viewed as left $A\#G$-module by

$$(a \cdot f)(g') := a \cdot f(g'),$$
$$(g \cdot f)(g') := g \cdot f(g'g)$$

for $a \in A$, $g, g' \in G$ and $f \in F_A^G(M)$ where, on the right hand side, the dot stands for the $A\#G$-action on $M$ (from now on we leave out the dot from the notations). The left $\mathbb{C}[G]$-action on $F_A^G(M)$, commuting with the above $A\#G$-action, is defined by

$$(\mu(g)f)(g') := f(g^{-1}g')$$

for $f \in F_A^G(M)$ and $g, g' \in G$. If $\phi$ is a morphism of $A\#G$-modules then $F_M^G(\phi)$ is set to be $(F_A^G(\phi)f)(g) := \phi(f(g))$.

Let $M$ be a left $A\#G$-module. Then we have a linear isomorphism of $A\#G$-modules

$$F_A^G(M)^{\mu(G)} \xrightarrow{\sim} M, \qquad f \mapsto f(e), \tag{5.11}$$

where $e \in G$ is the identity element (we could as well evaluate $f$ at any other element $g \in G$). We will freely use this identification in the remainder of this subsection. If $M$ is a $(A\#G, \mathbb{C}[G])$-bimodule, then we write $M^G$ for the subspace of $G$-invariants in $M$, with the $G$-action coming from the action of $A\#G$ on $M$.

**Lemma 5.12.** *Let $A$ be a unital associative $\mathbb{C}$-algebra endowed with an action of a group $G$ by unital algebra automorphisms. Let $M$ be a left $A\#G$-module.*

**(i)** *We have a linear isomorphism*

$$\Xi_M : M \xrightarrow{\sim} F_A^G(M)^G$$

*defined by* $(\Xi_M m)(g) := g^{-1}m$ *for* $g \in G$ *and* $m \in M$. *Furthermore,* $\Xi_M^{-1}(f) = f(e)$ *for* $f \in F_A^G(M)^G$.

**(ii)** *For* $m \in M$ *and* $g \in G$ *we have* $\Xi_M(gm) = \mu(g)(\Xi_M m)$. *In particular,* $\Xi_M|_{M^G} = \mathrm{id}$ *if we take the identification* (5.11) *into account.*

*Proof.* We write $\Xi = \Xi_M$ for the duration of the proof.
**(i)** Let $m \in M$ and $g, g' \in G$, then

$$\begin{aligned}\big(g'(\Xi m)\big)(g) &= g'\big((\Xi m)(gg')\big)\\ &= g'(g'^{-1}g^{-1})m\\ &= g^{-1}m\\ &= (\Xi m)(g).\end{aligned}$$

This shows that $\Xi m \in F_A^G(M)$ is $G$-invariant.

Define now $\Xi' : F_A^G(M)^G \to M$ by $\Xi'(f) = f(e)$. Note that $\Xi' \circ \Xi = \mathrm{id}$. On the other hand, if $f \in F_A^G(M)^G$, then for $g \in G$,

$$\Xi(\Xi'(f))(g) = g^{-1}(f(e)) = g^{-1}g(f(g)) = f(g),$$

where we used that $f$ is $G$-invariant in the second equality.
**(ii)** Let $m \in M$ and $g, g' \in G$. Then

$$(\Xi(gm))(g') = g'^{-1}gm = (\mu(g)(\Xi m))(g').$$

The last statement of **(ii)** is obvious. □

Since $W_0$ acts by conjugation on the subalgebra $\mathbb{D}_{\sigma,\nabla}^{k,q}$ of $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$, we have an isomorphism

$$\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W \simeq \mathbb{D}_{\sigma,\nabla}^{k,q}\#W_0$$

of algebras, cf. Proposition 5.5. We can thus apply the above constructions with $A = \mathbb{D}_{\sigma,\nabla}^{k,q}$ the algebra of $q$-difference operators viewed as a $G = W_0$-module algebra.

If $N$ is a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module, then we write

$$\widehat{N} := F_{\mathbb{D}_{\sigma,\nabla}^{k,q}}^{W_0}(N)$$

for the corresponding $(\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W, \mathbb{C}[W_0])$-bimodule. Concretely, $\widehat{N}$ consists of the space of functions $f : W_0 \to N$, with actions

$$\begin{aligned}(af)(w') &:= a\big(f(w')\big),\\ (wf)(w') &:= w\big(f(w'w)\big),\\ (\mu(w)f)(w') &:= f(w^{-1}w')\end{aligned}$$

for $f \in \widehat{N}$, $a \in \mathbb{D}_{\sigma,\nabla}^{k,q}$ and $w, w' \in W_0$.

Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module and $M$ a left $H(k)$-module. Then $\Gamma_L^{k,q}(M)_\nabla$ is a $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module via the algebra homomorphism

$$\nabla^{k,q} : \mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W \to \mathcal{A}^{k,q}$$

(see Corollary 3.7). Hence we can form the corresponding $\big(\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_q W, \mathbb{C}[W_0]\big)$-bimodule $\widehat{\Gamma^{k,q}_L(M)}_\nabla$. On the other hand, we have the left $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_q W$-module $\Gamma^{k,q}_{\widehat{L}}\big(M\big)_\nabla$, with the action again obtained from the algebra map $\nabla^{k,q}$. The $\mu(W_0)$-action on $\widehat{L}$ naturally extends to an $W_0$-action on $\Gamma^{k,q}_{\widehat{L}}\big(M\big)_\nabla$ (using $\Gamma^{k,q}_{\widehat{L}}\big(M\big) = \widehat{L}\otimes M$ as vector spaces, it is the $\mu(W_0)$-action on the first tensor leg). We will use the same notation $\mu$ for this extended action. With these two actions, $\Gamma^{k,q}_{\widehat{L}}\big(M\big)_\nabla$ becomes a $(\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_q W, \mathbb{C}[W_0])$-bimodule. Using (3.10), in particular using that $\nabla^{k,q}(\mathbb{D}^{k,q}_{\sigma,\nabla}) \subseteq \mathbb{D}^{k,q}_{\sigma,\nabla}\otimes_{\mathbb{C}} H(k)$, it follows that the canonical complex linear isomorphism

$$\widehat{\Gamma^{k,q}_L(M)}_\nabla \simeq \Gamma^{k,q}_{\widehat{L}}\big(M\big)_\nabla$$

is an isomorphism of $(\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_q W, \mathbb{C}[W_0])$-bimodules. We will use this isomorphism to identify $\widehat{\Gamma^{k,q}_L(M)}_\nabla$ and $\Gamma^{k,q}_{\widehat{L}}\big(M\big)_\nabla$ in the remainder of the text. Lemma 5.12 then gives

**Corollary 5.13.** *Let $L$ be a left $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_q W$-module and $M$ a left $H(k)$-module.*

**(i)** *We have a linear isomorphism*

$$\Xi : \Gamma^{k,q}_L(M)_\nabla \xrightarrow{\sim} \Gamma^{k,q}_{\widehat{L}}(M)^{W_0}_\nabla$$

*defined by $(\Xi f)(w) := \nabla^{k,q}(w^{-1})f$ for $w\in W_0$ and $f\in\Gamma_L(M)$. Furthermore, $\Xi^{-1}(h) = h(e)$ for $h\in\Gamma^{k,q}_{\widehat{L}}(M)^{W_0}_\nabla$. The map $\Xi$ restricts to a linear isomorphism*

$$\Gamma^{k,q}_L(M)^{\tau(P^\vee)}_\nabla \xrightarrow{\sim} \Gamma^{k,q}_{\widehat{L}}(M)^{W}_\nabla.$$

**(ii)** *For $f\in\Gamma^{k,q}_L(M)_\nabla$ and $w\in W_0$ we have $\Xi(\nabla^{k,q}(w)f) = \mu(w)(\Xi f)$. In particular, $\Xi|_{\Gamma^{k,q}_L(M)^{W_0}_\nabla} = \mathrm{id}$ if we take the identification (5.11) into account.*

*Proof.* Apply Lemma 5.12 to the $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_q W$-module $\Gamma^{k,q}_L(M)_\nabla$ and set $\Xi = \Xi_{\Gamma^{k,q}_L(M)_\nabla}$. This gives all the results besides the second statement of **(i)**, which though follows by a direct computation. □

The corollary can be used to formulate nonsymmetric versions of Theorem 4.9 and of Proposition 5.10. We start with the nonsymmetric version of Theorem 4.9.

**Corollary 5.14.** *Let $\gamma\in T^{k^{\pm1}}_I$. Let $L$ be a left $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_q W$-module. We have a linear isomorphism*

$$\eta_\pm : \widehat{L}^{I^*,\pm}_\pi[\overline{w_0}\gamma^{-1}] \xrightarrow{\sim} \Gamma^{k,q}_L\big(M^{k,\pm,I}(\gamma)\big)^{\tau(P^\vee)}_\nabla,$$

*given by*

$$\eta_\pm(\phi) := \sum_{w\in W_0^I}\big(\pi^{k^{-1},q}(T_{w\overline{w_0}^{-1}})\phi\big)(e)\otimes v^{k,\pm,I}_w(\gamma), \qquad \phi\in\widehat{L}^{I^*,\pm}_\pi[\overline{w_0}\gamma^{-1}].$$

*Furthermore, $\widehat{L}^{I^*,\pm}_\pi[\overline{w_0}\gamma^{-1}]\subseteq\widehat{L}$ is a $\mu(W_0)$-submodule, and, with the identification (5.11), the map $\eta_{\widehat{L}^{I^*,\pm}_\pi[\overline{w_0}\gamma^{-1}]^{\mu(W_0)}}$ coincides with the isomorphism*

$$L^{I^*,\pm}_\pi[\overline{w_0}\gamma^{-1}] \xrightarrow{\sim} \Gamma^{k,q}_L\big(M^{k,\pm,I}(\gamma)\big)^{W}_\nabla$$

*from Theorem 4.9, given by $\phi\mapsto\psi_\phi$ (see (4.6)).*

*Proof.* The first statement follows from the chain of isomorphisms

$$\widehat{L}_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}] \xrightarrow{\sim} \Gamma_{\widehat{L}}^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W \xrightarrow{\sim} \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{\tau(P^\vee)},$$

with the first isomorphism given by

$$\widehat{L}_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}] \ni \phi \mapsto \psi_\phi = \sum_{w\in W_0^I} \pi^{k^{-1},q}(T_{w\overline{w_0}^{-1}})\phi \otimes v_w^{k,\pm,I}(\gamma)$$

(cf. Theorem 4.9), and the second isomorphism given by $\Xi^{-1}$, which maps $f \in \Gamma_{\widehat{L}}^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^W$ to $f(e) \in \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{\tau(P^\vee)}$.

It is clear that $\widehat{L}_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}] \subseteq \widehat{L}$ is a $\mu(W_0)$-submodule. Let $w' \in W_0$ and $\phi \in \widehat{L}_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$. Then

$$\eta_\pm(\mu(w')\phi) = \sum_{w\in W_0^I} \big(\pi^{k^{-1},q}(T_{w\overline{w_0}^{-1}})\phi\big)(w'^{-1}) \otimes v_w^{k,\pm,I}(\gamma).$$

Thus, with the identification $\widehat{L}^{\mu(W_0)} \simeq L$ as $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-modules (see (5.11)), we get for $\phi \in \widehat{L}_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]^{\mu(W_0)} \simeq L_\pi^{I^*,\pm}[\overline{w_0}\gamma^{-1}]$,

$$\eta_\pm(\phi) = \sum_{w\in W_0^I} \pi^{k^{-1},q}(T_{w\overline{w_0}^{-1}})\phi \otimes v_w^{k,\pm,I}(\gamma) = \psi_\phi.$$

□

Similarly we are now going to prove the nonsymmetric version of Proposition 5.10 and Proposition 5.3. We need the following prepatory lemma. Recall that $W_{0,\pm} = \{w_\pm\}_{w\in W_0} \subset \big(\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW\big)^\times$, with $w_\pm$ given by (4.2).

**Lemma 5.15.** *Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module.*
**(a)** *We have a linear isomorphism*

$$\Xi_L^\pm : L \xrightarrow{\sim} \widehat{L}^{W_{0,\pm}}$$

*defined by $(\Xi_L^\pm\phi)(w) := w_\pm^{-1}\phi$ for $w \in W_0$ and $\phi \in L$. Furthermore, $\big(\Xi_L^\pm\big)^{-1}(h) = h(e)$ for $h \in \widehat{L}^{W_{0,\pm}}$. The map $\Xi_L^\pm$ restricts to a linear isomorphism*

$$\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma^{-1}) \xrightarrow{\sim} \mathrm{SpM}_{\widehat{L}}^{k,q,\pm}(W_0\gamma^{-1})^{W_{0,\pm}}.$$

**(b)** *For $\phi \in L$ and $w \in W_0$ we have $\Xi_L^\pm(w_\pm\phi) = \mu(w)(\Xi_L^\pm\phi)$. In particular, $\Xi_L^\pm|_{L^{W_{0,\pm}}} = \mathrm{id}$ if we take the identification* (5.11) *into account.*

*Proof.* This is a straightforward adjust of the proof of Corollary 5.13. The second part of **(a)** uses the fact that the $D_f^{k,q,\pm}$ $(f \in \mathbb{C}[T]^{W_0})$ are $W_{0,\pm}$-equivariant, cf. Proposition 5.5. □

**Theorem 5.16.** *Let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_qW$-module.*
**(a)** *Let $\gamma \in T_I^{k^{\pm 1}}$. The Cherednik-Matsuo map $\xi_{L,\gamma}^{k,q,\pm,I}$ (see Definition 5.2) restricts to a linear map $W_0$-equivariant (in the sense of* (5.6)*) linear map*

$$\xi_{L,\gamma}^{k,q,\pm,I} : \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{\tau(P^\vee)} \to \mathrm{SpM}_L^{k,q,\pm}(W_0\gamma^{-1}).$$

*The map is injective if* $\gamma \in T^{k^{\pm 1}}_{I,reg}$ *(see* (5.2)*).*

**(b)** *Let* $\gamma \in T$ *such that* $k^2_{\alpha^\vee} \neq \gamma^{\alpha^\vee} \neq 1$ *for all* $\alpha \in R_0$*. Then* $\xi^{k,q,\pm}_{L,\gamma}$ *restricts to a* $W_0$*-equivariant (in the sense of* (5.6)*) linear isomorphism*

$$\xi^{k,q,\pm}_{L,\gamma} : \Gamma^{k,q}_L\big(M^k(\gamma)\big)^{\tau(P^\vee)}_\nabla \xrightarrow{\sim} \mathrm{SpM}^{k,q,\pm}_L(W_0\gamma^{-1}).$$

*Proof.* Let $\gamma \in T^{k^{\pm 1}}_I$ and let

$$\xi : \Gamma^{k,q}_L\big(M^{k,\pm,I}(\gamma)\big)^{\tau(P^\vee)}_\nabla \to \mathrm{SpM}^{k,q,\pm}_L(W_0\gamma^{-1})$$

be the linear map such that the following diagram

$$\begin{tikzcd}
\Gamma^{k,q}_L\big(M^{k,\pm,I}(\gamma)\big)^{\tau(P^\vee)}_\nabla \arrow[r, "\sim", "\Xi"'] \arrow[d, "\xi"] & \Gamma^{k,q}_{\widehat{L}}\big(M^{k,\pm,I}(\gamma)\big)^W_\nabla \arrow[d, "\xi^{k,q,\pm,I}_{\widehat{L},\gamma}"] \\
\mathrm{SpM}^{k,q,\pm}_L(W_0\gamma^{-1}) \arrow[r, "\sim", "\Xi^\pm_L"'] & \mathrm{SpM}^{k,q,\pm}_{\widehat{L}}(W_0\gamma^{-1})^{W_{0,\pm}}
\end{tikzcd}$$

is commutative. In view of Proposition 5.10 (applied to the $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]\#_qW$-module $\widehat{L}$) it then suffices to show that

$$\xi = \xi^{k,q,\pm,I}_{L,\gamma}.$$

Let $\phi \in \Gamma^{k,q}_L\big(M^{k,\pm,I}(\gamma)\big)^{\tau(P^\vee)}_\nabla$ and write $\Xi(\phi) = \sum_{w\in W_0^I} \widehat{\psi}_w \otimes v^{k,\pm,I}_w(\gamma)$ with $\widehat{\psi}_w \in \widehat{L}$. Then by Corollary 5.14 and Lemma 5.15,

$$\begin{aligned}
\xi^{k,q,\pm,I}_{L,\gamma}(\phi) &= \xi^{k,q,\pm,I}_{L,\gamma}\big((\Xi\phi)(e)\big)\\
&= \sum_{w\in W_0^I} \epsilon^k_\pm(T_w)\widehat{\psi}_w(e)\\
&= \big(\xi^{k,q,\pm,I}_{\widehat{L},\gamma}(\Xi\phi)\big)(e)\\
&= \big(\Xi^\pm_L\big)^{-1}\xi^{k,q,\pm,I}_{\widehat{L},\gamma}\Xi(\phi)\\
&= \xi(\phi),
\end{aligned}$$

as desired. □

*Remark* 5.17. Cherednik used different methods in [4] to show that $\xi^{k,q,+}_{L,\gamma}$ defines a linear map $\xi^{k,q,+}_{L,\gamma} : \Gamma^{k,q}_L(M^k(\gamma))^{\tau(P^\vee)}_\nabla \to \mathrm{SpM}^{k,q,+}_L(W_0\gamma^{-1})$. The advantage of the present techniques is that they lead to precise conditions on $\gamma$ and $k$ to ensure that $\xi^{k,q,+}_{L,\gamma}$ is a linear isomorphism. Such properties of the map $\xi^{k,q,+}_{L,\gamma}$ (for special choices of $L$ and $q$) were also discussed in [4, Thm. 4.3]. It is though not clear to the author that the suggested proof of [4, Thm. 4.3] (based on classical techniques from, e.g., [15, Part I, Section 4.1]) works out in the present $q$-setup.

*Remark* 5.18. The classical analogue of Theorem 5.16**(b)** is due to Matsuo [25, Thm. 5.4.1] (in the symmetric (+) case) and Cherednik [3, Thm. 4.7]. The arguments leading to Theorem 5.16**(b)** is motivated by Opdam's [28, §3 ] approach to this classical correspondence. It is likely that Theorem 5.16**(b)** can be strenghened a bit, in the sense that the genericity conditions $k^2_{\alpha^\vee} \neq \gamma^{\alpha^\vee} \neq 1$ ($\alpha \in R_0$) can be weaked further (so that they match the genericity conditions in [25, Thm. 5.4.1], [3, Thm. 4.7] and [28, Cor. 3.12] for the classical correspondence). Note that the

sharpened genericity conditions in [28, Cor. 3.12], compared to e.g. the genericity conditions in [28, Cor. 3.11], are justified by the fact that the analogue of $\mathrm{SpM}_L^{k,q,+}(W_0\gamma^{-1})$ is known to be of dimension $\#W_0$; such a type of result is not known in the present setup, as far as we know.

*Remark* 5.19. Using the notations from Subsection 4.3, Theorem 5.16 holds true in the $\mathrm{GL}_m$-case. Here $P^\vee$ should be taken to be the lattice $\mathbb{Z}^m$, the complex torus is $T = (\mathbb{C}^\times)^m$ and $W_0 = S_m$. In the symmetric case $(+)$ the commuting $q$-difference operators $D_f^{k,q,+}$ ($f \in \mathbb{C}[T]^{S_m}$) were obtained for the first time by Ruijsenaars [32]. Concretely, for the elementary symmetric functions $e_i \in \mathbb{C}[T]^{S_m}$ $(1 \leq i \leq m)$ defined by

$$e_i(t) = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ \#I = i}} \prod_{j \in I} t_j,$$

we have the explicit expressions

$$D_{e_i}^{k,q,+} = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ \#I = i}} \left( \prod_{\substack{r \in I \\ s \notin I}} \frac{kt_r - k^{-1}t_s}{t_r - t_s} \right) \tau\big(\sum_{r \in I} \epsilon_r\big) \in \mathbb{C}[T]_{\sigma,\nabla}^{k,q} \#_q \mathbb{Z}^m,$$

see, e.g., [7, §1.3.5] and [19].

Recall the morphism (4.15) of $\mathbb{C}[W]$-modules, valid for $\gamma \in T_I^{k^{\pm 1}}$. Combined with Theorem 5.16 we obtain

**Corollary 5.20.** *Let $\gamma \in T_I^{k^{\pm 1}}$ and let $L$ be a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$-module. We have a commutative diagram of linear maps*

$$\begin{tikzcd}
\Gamma_L^{k,q}\big(M^k(\gamma)\big)_\nabla^{\tau(P^\vee)} \arrow[r] \arrow[d, "\xi_{L,\gamma}^{k,q,\pm}"] & \Gamma_L^{k,q}\big(M^{k,\pm,I}(\gamma)\big)_\nabla^{\tau(P^\vee)} \arrow[dl, "\xi_{L,\gamma}^{k,q,\pm,I}"] \\
\mathrm{SpM}_L^{k,q,\pm}(W_0\gamma^{-1}) &
\end{tikzcd}$$

*where the horizontal arrow is the restriction of the map (4.15) to $\Gamma_L^{k,q}(M^k(\gamma))_\nabla^{\tau(P^\vee)}$. The southwest arrow represents an injective map if $\gamma \in T_{I,reg}^{k^{\pm 1}}$ (see (5.2)).*

5.5. **An application.** The Cherednik-Matsuo correspondence (Theorem 5.16) with $L = \mathcal{M}(T)$ and $0 < |q| < 1$ is instrumental for quantum (noncompact) harmonic analysis; the author will discuss this in the forthcoming second part [35] of this paper. In this subsection we consider consequences of the Cherednik-Matsuo correspondence when $q = 1$; this case actually also deserves much more attention in view of the potential applications to spin chains and the Razumov-Stroganov conjectures, see, e.g., [31, 30, 12, 16]. We hope to return to this in more detail in future work.

For $f \in \mathbb{C}[T]^{W_0}$ we write the $q$-difference operator $D_f^{k,q,\pm} \in \mathbb{D}_{\sigma,\nabla}^{k,q}$ as

$$D_f^{k,q,\pm} = \sum_{\lambda \in P^\vee} u_{f,\lambda}^{k,q,\pm} \tau(\lambda)$$

with $u_{f,\lambda}^{k,q,\pm} \in \mathbb{C}_{\sigma,\nabla}^{k,q}[T]$.

The proof of the following lemma hinges on the theory of (anti)symmetric Macdonald polynomials (cf. Remark 5.9). Recall the definition (2.13) of $\delta_\pm^k \in T$.

**Lemma 5.21.** *Let $f \in \mathbb{C}[T]^{W_0}$. Then*
$$\sum_{\lambda\in P^\vee} u^{k,1,\pm}_{f,\lambda} = f\big(\delta^k_+\big)$$
*as identity in $\mathbb{C}^{k,1}_{\sigma,\nabla}[T]$.*

*Proof.* We use the notations from Remark 5.9. Suppose for the moment that we have fixed $q$ and $k$ satisfying the generic conditions as in Remark 5.9. Then it is known that, up to a nonzero constant,
$$P^{(+)}_0(k^{-1},q)=1, \qquad P^{(-)}_{\rho^\vee}(k^{-1},q)=G^{k,-},$$
where $1\in\mathbb{C}[T]$ is the constant function one, cf., e.g., [23, (5.8.10)] for the second equality. Thus for all possible values of $q$ and $k$,
$$\begin{aligned}1&\in \mathrm{SpM}^{k,q,+}_L(W_0\gamma_0(k^{-1},q)^{-1})^{W_{0,+}},\\ G^{k,-}&\in \mathrm{SpM}^{k,q,-}_L(W_0\gamma_{\rho^\vee}(k^{-1},q)^{-1})^{W_{0,-}}\end{aligned} \tag{5.12}$$
with $L=\mathbb{C}^{k,q}_{\sigma,\nabla}[T]$. Since
$$\gamma_0(k^{-1},q)^{-1}=\prod_{\alpha\in R_0^+} k_{\alpha^\vee}^{-\alpha}=w_0(\delta^k_+)$$
(independent of $q$) and
$$\gamma_{\rho^\vee}(k^{-1},q)^{-1}=q^{-\rho^\vee}\prod_{\alpha\in R_0^+}k^{\alpha}_{\alpha^\vee}=q^{-\rho^\vee}\delta^k_+,$$
we get for all $f\in\mathbb{C}[T]^{W_0}$,
$$\sum_{\lambda\in P^\vee}u^{k,q,+}_{f,\lambda}=D^{k,q,+}_f(1)=f\Big(\gamma_0(k^{-1},q)^{-1}\Big)=f\big(\delta^k_+\big)$$
in $\mathbb{C}^{k,q}_{\sigma,\nabla}[T]$ (this is valid for all $q\in\mathbb{C}^\times$, in particular for $q=1$), as well as
$$\sum_{\lambda\in P^\vee}u^{k,1,-}_{f,\lambda}=\big(G^{k,-}\big)^{-1}D^{k,1,-}_f\big(G^{k,-}\big)=f\Big(\gamma_{\rho^\vee}(k^{-1},1)^{-1}\Big)=f\big(\delta^k_+\big)$$
in $\mathbb{C}^{k,1}_{\sigma,\nabla}[T]$. □

**Proposition 5.22.** *Let $L$ be a left $\mathbb{C}^{k,1}_{\sigma,\nabla}[T]\#W_0$ module. Turn $L$ into a $\mathbb{C}^{k,1}_{\sigma,\nabla}[T]\#_1W$ module by letting $\tau(P^\vee)$ act trivially on $L$. Suppose that $\gamma\in T^{k^{\pm1}}_{I,reg}$ (see (5.2)) and $\gamma\notin W_0(\delta^k_+)$. Then $\Gamma^{k,1}_L\big(M^{k,\pm,I}(\gamma)\big)^{\tau(P^\vee)}_\nabla=\{0\}$.*

*Proof.* Suppose that $\gamma\in T^{k^{\pm1}}_{I,reg}$ and $\gamma\notin W_0(\delta^k_+)$. In view of Theorem 5.16**(a)** it suffices to prove that $\mathrm{SpM}^{k,1,\pm}_L(W_0\gamma^{-1})=\{0\}$.

Since $L=L^{\tau(P^\vee)}$, the previous lemma shows that
$$\mathrm{SpM}^{k,1,\pm}_L(W_0\gamma^{-1})=\{\phi\in L\ \mid\ f\big(\delta^k_+\big)\phi=f(\gamma^{-1})\phi\qquad\forall f\in\mathbb{C}[T]^{W_0}\}.$$
Consequently
$$\mathrm{SpM}^{k,1,\pm}_L(W_0\gamma^{-1})=\begin{cases}L & \text{if }\ \gamma\in W_0\big(\delta^k_+\big),\\ \{0\} & \text{if }\ \gamma\notin W_0\big(\delta^k_+\big),\end{cases}$$
hence the result. □

*Remark* 5.23. Let $L$ be field and a left $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$ module such that $W$ acts by field automorphisms on $L$. Let $M$ be a finite dimensional $H(k)$-module over $\mathbb{C}$. Then it can be shown that

$$\mathrm{Dim}_{L^{\tau(P^\vee)}}\big(\Gamma_L^{k,q}(M)_{\nabla}^{\tau(P^\vee)}\big) \leq \mathrm{Dim}_{\mathbb{C}}(M). \tag{5.13}$$

Since the quantum affine KZ equations are holonomic it is natural to expect equality in (5.13). This is for instance the case if $0 < |q| < 1$ and if $L = \mathcal{M}(T)$ with the left action of $\mathbb{C}_{\sigma,\nabla}^{k,q}[T]\#_q W$ on $\mathcal{M}(T)$ by $q$-difference reflection operators (see [11]). In general equality does not hold true though, as Proposition 5.22 shows.

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KdV Institute for Mathematics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands.

*E-mail address*: j.v.stokman@uva.nl